Multi-depot routing with split deliveries: Models and a branch-and-cut algorithm

Luis Gouveia[1], Markus Leitner[2] and Mario Ruthmair[3,4,2]

[1]Universidade de Lisboa, Faculdade de Ciências, Departamento de Estatística e Investigação Operacional, Lisbon, Portugal. legouveia@fc.ul.pt

[2]Department of Operations Analytics, Vrije Universiteit Amsterdam, Netherlands. m.leitner@vu.nl

[3]University of Vienna, Department of Statistics and Operations Research, Vienna, Austria. mario.ruthmair@univie.ac.at

[4]University of Klagenfurt, Department of Mathematics, Klagenfurt, Austria.

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Abstract

We study the split-delivery multi-depot vehicle routing problem (MDSDVRP) which combines the advantages and potential cost-savings of multiple depots and split-deliveries and develop the first exact algorithm for this problem. We propose an integer programming formulation using a comparably small number of decision variables and several sets of valid inequalities. These inequalities focus on ensuring the vehicles' capacity limits and that vehicles return to their initial depot. As we show that the new constraints do not guarantee these aspects our branch-and-cut framework also includes an efficient feasibility check for candidate solutions and explicit feasibility cuts. The algorithm is tested on the MDSDVRP and two well-known special cases, the split-delivery vehicle routing problem (SDVRP) and the multi-depot traveling salesman problem (MDTSP). The results show that the new inequalities tighten the linear programming relaxation, increase the performance of the branch-and-cut algorithm, and reduce the number of required feasibility cuts. We report the first proven optimal results for the MDSDVRP and show that our algorithm significantly outperforms the state-of-the-art for the MDTSP while being competitive on the SDVRP. For the latter, 20 instances are solved for the first time and new best primal and dual bounds are found for others.

Keywords: vehicle routing, multi-depot, split-delivery, branch-and-cut, valid inequalities

1 Introduction

Multi-depot routing problems study the distribution of goods from a set of locations to customers. This class of problems and its location-routing counterpart in which location decision on depots need to be made simultaneously have received significant attention since the early works by Laporte et al. (1984, 1986, 1988). Besides being versatile models relevant in many real-world settings, variants of the *multi-depot capacitated vehicle routing problem (MDCVRP)* (Laporte et al., 1984) or the *capacitated location routing problem (CLRP)* (Laporte et al., 1986) also arise as subproblems in other, highly relevant problems such as the *two-echelon capacitated vehicle routing problem* (Perboli et al., 2011). Recent generic branch-price-and-cut based solution frameworks for vehicle routing problems (Baldacci and Mingozzi, 2009; Pessoa et al., 2020) have also been applied to the MDCVRP and significantly increased the size of instances that can be solved. A parallel stream of papers focuses on the study of inequalities that prevent vehicles from ending their trip at a depot different from their initial one, see, e.g., Belenguer et al. (2011); Benavent and Martínez (2013); Bektaş et al. (2017, 2020); uit het Broek et al. (2021). Ensuring the latter aspect is the main additional challenge imposed by multi-depot routing problems when compared to their single-depot counterparts.

A common characteristic of these problems is that each customer's demand is served by a single visit of one vehicle. Dror and Trudeau (1989, 1990) observed that the total travel times may be significantly reduced by relaxing the latter condition in single-depot routing problems. Indeed, Archetti et al. (2006a) showed that this reduction can be up to 50% in the *split-delivery vehicle routing problem (SDVRP)*. As for other routing problems, the performance of exact, integer programming based algorithms for the SDVRP and its variants has recently been improved significantly, see Archetti et al. (2014); Bianchessi and Irnich (2019); Bianchessi et al. (2019); Munari and Savelsbergh (2019); Ozbaygin et al. (2018). In contrast to routing problems without split-deliveries, most state-of-the-art methods for SDVRP variants are based on *branch-and-cut (B&C)*.

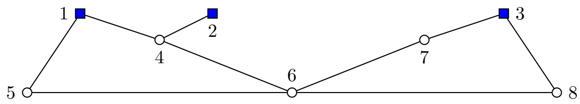

Figure 1: A solution to the MDSDVRP.

Despite its practical relevance and potentially huge cost savings, the combination of split-deliveries and multiple-depots has been considered only recently. Gulczynski et al. (2011) introduced the *multi-depot split-delivery vehicle routing problem (MDSDVRP)* and report significant travel-time reductions when comparing their heuristic solutions to those of a problem variant in which customers are visited only once. Ray et al. (2014) proposed a heuristic approach for the MDSDVRP and were the first to propose an ILP formulation for the MDSDVRP which was, however, not empirically tested in their computational study. Fernández et al. (2018) introduce and study the shared customer collaboration VRP which is a variant of the MDSDVRP in which a customer's demand cannot be split arbitrarily among different vehicles (carriers) but a predefined set of splitting options is given. Extended formulations based on vehicle-indexed and load-indexed variables as well as exact, B&C based solution algorithms are proposed.

The MDSDVRP is defined on an undirected graph $G = (V, E)$ whose node set is partitioned into a set of depots $D \subset V$ and a set of customers $C = V \setminus D$. A fleet of K homogeneous vehicles each with capacity $Q \in \mathbb{N}$ is used to satisfy demands $q_j \in \mathbb{N}$ of customers $j \in C$. The edge set $E = E_D \cup E_C$ contains depot edges $E_D = \{\{d, j\} \mid d \in D, j \in C\}$ and customer edges $E_C = \{\{i, j\} \mid \{i, j\} \subseteq C, i \neq j\}$ with associated travel costs $c_e \geq 0$, $\forall e \in E$, which satisfy the triangle inequality. The aim is to identify a set of vehicle routes minimizing the total travel costs while satisfying all customer demands, respecting vehicle capacities, and ensuring that each vehicle returns to its initial depot. Figure 1 shows a solution of an instance with $D = \{1, 2, 3\}$ and $C = \{4, 5, 6, 7, 8\}$ for which we assume that all capacity constraints are met. In this solution, customer 6 is served via two routes starting at depots 1 and 3 while customer 4 is served via a single-customer route from depot 2 and the route starting from depot 1. This example illustrates that in contrast to multi-depot routing problems without split deliveries, paths between different depots are allowed in the MDSDVRP when vehicles located at different depots visit the same customer.

1.1 Scientific contribution and outline

This paper is the first study of ILP models and exact solutions methods for multi-depot routing problems with split-deliveries. Its main contributions can be summarized as follows:

- A generic ILP formulation for the MDSDVRP using a small number of variables. The formulation allows to independently study the main challenges of the MDSDVRP which are to ensure (i) the vehicles' capacity limits and (ii) that each vehicle returns to its initial depot.
- New capacity constraints exploiting single-customer route variables.
- New depot-consistency constraints aiming to ensure that vehicles return to their initial depot.
- A proof that despite the new families of inequalities, all considered formulations model only a relaxation of the MDSDVRP. This relaxation does neither ensure the vehicles' capacity limits nor that each vehicle returns to its initial depot.
- Development of an algorithmic framework that is applicable to the MDSDVRP and several variants of it. Besides the newly proposed sets of inequalities, this framework is based on the efficient identification of infeasible solutions that are cut-off via additional valid inequalities.
- A fast heuristic used to initialize the B&C which is competitive to other MDSDVRP heuristics.
- State-of-the-art computational results for the MDSDVRP, the SDVRP, and the *multi-depot traveling salesman problem (MDTSP)*.

This paper is organized as follows. In the remainder of this section, we discuss properties of optimal solutions and introduce required notation. Section 2 presents the generic ILP model and its adaptation to special cases. Section 3 introduces several new inequalities that exploit the inclusion of single-customer route variables in our model. Sections 4 and 5 propose and compare several new families of inequalities that aim to ensure the vehicles' capacities and that each vehicle returns to its initial depot, respectively. In these sections, we also show that each of these two requirements can only be partially ensured by the constraints discussed in this paper. Details of our algorithmic framework are given in Section 6. Section 7 discusses the results and findings of our computational study, before final conclusions and potential future research directions are given in Section 8. The proofs of all theoretical results and detailed computational results are given in the appendix.

1.2 Solution properties

In this section, we state properties that hold for optimal solutions to the MDSDVRP and all special cases discussed in this article. These properties assume that the travel costs satisfy the triangle inequality and will be relevant for the development of the formulations and methods discussed in the following sections. Properties 1, 2, and 3 are known for the SDVRP (Archetti et al., 2006a; Desaulniers, 2010; Dror and Trudeau, 1989, 1990; Dror et al., 1994). They follow from a result by Dror and Trudeau (1990) which states that there always exists an optimal solution to the SDVRP that does not contain so-called k-split cycles. The proof given in Dror and Trudeau (1990) is based only on exchange arguments between vehicles and all these arguments carry over to the MDSDVRP.

Property 1. *There exists an optimal solution to the MDSDVRP such that the subgraph of this solution induced by the customer set C is acyclic.*

Property 2. *There exists an optimal solution to the MDSDVRP in which each edge between two customers is traversed at most once.*

Property 3. *There exists an optimal solution to the MDSDVRP in which the total number of splits is strictly smaller than the number of routes.*

In contrast to the previous ones, Properties 4, 5, and 6 are new to our knowledge. Property 5 follows from Property 4. This can be seen by iteratively applying the arguments used in the proof of Property 4 (see the appendix) for solution subgraphs in which a route with at most one split node is removed. Property 6 is a stronger version of Property 4 that follows from Property 5 as acyclic graphs contain at least two leaves and since split nodes are contained in at least two routes.

Property 4. *There exists an optimal solution to the MDSDVRP in which at least one route contains at most one split node.*

Property 5. *Let $\mathcal{G}(S) = (\mathcal{R} \cup \mathcal{S}, \mathcal{E})$ be the bipartite graph corresponding to some MDSDVRP solution S where $\mathcal{R}$ and $\mathcal{S}$ contain one node for each route and split node in S, respectively. Edge $\{r, s\}$ is included in $\mathcal{E}$ if route $r \in \mathcal{R}$ visits split node $s \in \mathcal{S}$. There exists an optimal solution S to the MDSDVRP such that $\mathcal{G}(S)$ is acyclic.*

Property 6. *If at least one optimal solution to the MDSDVRP consists of more than one route, then there exists an optimal solution in which at least two routes contain at most one split node.*

1.3 Notation

In the remainder of this article, we will use notations $\delta(S) = \{\{i,j\} \in E \mid i \in S, j \notin S\}$ and $\delta_D(S) = \{\{i,j\} \in E_D \mid i \in S, j \notin S\}$ to denote the set of edges and depot edges, respectively, with one incident node in set $S \subset V$. Similarly, $\delta(S,T) = \{\{i,j\} \in E \mid i \in S, j \in T\}$ is the set of edges between node sets $S \subset V$ and $T \subseteq V \setminus S$ while $E(S) = \{\{i,j\} \in E \mid \{i,j\} \subseteq S\}$ is the set of edges in set $S \subseteq V$. For simplicity, we will omit set braces if a set contains only one node. For a set of variables X defined over some ground set $\mathcal{M}$ and a set $M \subseteq \mathcal{M}$, we use compact notation $X(M) = \sum_{i \in M} X_i$. Finally, $\bar{v}$ will be used to denote the value of some variable v in a (candidate) solution.

2 Basic model and problem variants

This section presents a generic ILP formulation used as a basis for studying different sets of inequalities presented in the following sections. Adaptations for variants of the MDSDVRP are discussed at the end of the section. The generic model uses three sets of variables: (i) edge decision variables $x_e \in \mathbb{N}_0$, $e \in E_D$, and $x_e \in \{0,1\}$, $e \in E_C$, (ii) integer variables $z_i \in \mathbb{N}$, $i \in V$, and (iii) integer variables $y_i \in \{0, 1, \ldots d_i/Q\rceil\}$, $i \in C$. Variables x_e indicate the number of vehicles that traverse a depot or customer edge. As mentioned above, only edges incident to a depot may be traversed by more than one vehicle in an optimal solution. Variables z_i indicate the number of vehicles that visit a node. Finally, variables y_i, indicate the number of single-customer routes to a customer. Observe that (in an optimal solution) a single-customer route to customer $i \in C$ is performed from the closest depot $d = \text{argmin}_{g \in D} c_{gi}$ and has costs $c_i = 2c_{di}$. One closest depot is chosen arbitrarily in case of draws. Notation $N_{\text{L}}(d) \subseteq C$ will be used to indicate the set of customers that may be served by a single-customer route from depot $d \in D$.

$$\min \sum_{e \in E} c_e x_e + \sum_{i \in C} c_i y_i \tag{1a}$$

$$x(\delta(i)) + 2y_i = 2z_i \qquad \forall i \in C \tag{1b}$$

$$x(\delta(d)) + 2y(N_{\text{L}}(d)) = 2z_d \qquad \forall d \in D \tag{1c}$$

$$\sum_{d \in D} z_d \leq K \tag{1d}$$

(a) (b)

Figure 2: Candidate solutions that are not valid for the MDSDVRP and which do not violate any (a) capacity and (b) depot-consistency constraints considered in this article.

$$x(E(S)) \leq |S| - 1 \qquad \forall S \subseteq C \tag{1e}$$

$$\text{capacity constraints} \tag{1f}$$

$$\text{depot-consistency constraints} \tag{1g}$$

$$x_e \in \{0, 1, \ldots, \min\{K, d_i\}\} \qquad \forall e = \{d, i\} \in E_D \tag{1h}$$

$$x_e \in \{0, 1\} \qquad \forall e \in E_C \tag{1i}$$

$$y_i \in \{0, 1, \ldots, \lceil d_i/Q \rceil\} \qquad \forall i \in C \tag{1j}$$

$$z_i \in \{\lceil q_i/Q \rceil, ..., \min\{K, q_i\}\} \qquad \forall i \in V \tag{1k}$$

The objective function (1a) minimizes the total travel costs. Degree constraints (1b) and (1c) link the numbers of edge traversals and single-customer routes to the number of vehicles visiting a node. Inequality (1d) establishes the upper bound K on the total number of vehicles. As discussed in Section 1.2, the validity of subtour elimination constraints (1e) follows from a result by Dror and Trudeau (1990). Different families of inequalities replacing constraints (1f) are discussed in Section 4. Their goal is to ensure that the correct quantities are delivered to all customers while respecting the capacities of all vehicles. Similarly, different sets of linear inequalities replacing generic constraints (1g) will be discussed in Section 5. These *depot-consistency constraints* aim to ensure that each vehicle returns to the depot where it started from.

As indicated earlier, this study focuses on capacity constraints and depot-consistency inequalities that do not introduce additional variables. As shown later, all considered variants of formulation (1) that use different sets of constraints in place of (1f) and (1g) will model only a relaxation of the MDSDVRP and other variants of multi-depot split-delivery routing problems, see Sections 4 and 5. A similar result is well-known for the SDVRP with respect to the capacity-constraints (Archetti et al., 2014) and we will show an analogous result with respect to the depot-consistency constraints. In line with state-of-the-art methods for (variants of) the SDVRP (Archetti et al., 2014; Bianchessi and Irnich, 2019; Bianchessi et al., 2019), we will exclude infeasible candidate solutions that do not violate other constraints via inequalities (2) that are added dynamically in a branch-and-cut fashion. For each such candidate solution with associated variable values $\bar{x}_e$, $\forall e \in E$, and $\bar{y}_i$, $\forall i \in C$, inequality (2) states that at least one edge or single-customer route not part of this solution has to be used, cf. Archetti et al. (2014); Bianchessi and Irnich (2019).

$$x(\{e \in E : \bar{x}_e = 0\}) + y(\{i \in C : \bar{y}_i = 0\}) \geq 1 \tag{2}$$

The need for these inequalities is illustrated by the two candidate solutions corresponding to the graphs given in Figure 2a and Figure 2b with $D = \{1\}$ and $D = \{1, 2, 3\}$, respectively. An edge $e = \{u, v\}$ between two nodes indicates that $\bar{x}_e = 1$. Since all node degrees are even and the subgraphs induced by customer set C are acyclic, only capacity or depot-consistency constraints may be violated. Indeed, the solution given in Figure 2a is infeasible when assuming a vehicle capacity of $Q = 7$, and customer demands $q_6 = 6$, and $q_i = 2$, $i \in C \setminus \{6\}$. Since each potential route visiting customer 6 must also visit two additional customers (5 and one from set $\{2, 3, 4\}$) the demand served by such a route would exceed the vehicle capacity. We will show in Section 4 that this candidate solution satisfies all capacity constraints considered in this article, see Theorem 2. Next, consider the candidate solution shown in Figure 2b and assume that the capacity limits of all vehicles are met. To see why this candidate solution is infeasible, observe that the route containing edge $\{3, 7\}$ must also contain edges $\{5, 6\}$ and $\{6, 8\}$. Therefore, edge $\{1, 6\}$ can not be included in a valid route since the only still unused edge incident to node 6 is $\{2, 6\}$ leading to the path $(1, 6, 2)$ connecting two different depots. Even though several families of depot-consistency constraints are proposed in Section 5, Theorem 5 will show that this solution does not violate any of them.

2.1 Problem variants

Several variants of the MDSDVRP can be modeled by minor modifications of formulation (1). This enables the development of a generic B&C algorithm that will be described in Section 6 and whose efficiency will be demonstrated in Section 7 for all considered variants. In addition to the MDSDVRP, our computational study considers the SDVRP and the MDTSP (Bektaş et al., 2017, 2020; Benavent and Martínez, 2013) focusing on split-deliveries and multiple depots, respectively. The basic model for the SDVRP is obtained from formulation (1) by removing constraints (1g). For the MDTSP, we remove capacity constraints (1f), variables z_i, $\forall i \in V$, and degree constraints (1c) for all depots. Furthermore, the upper bounds of all integer variables are set to one and degree constraints (1b) are replaced by $x(\delta(i)) + 2y_i = 2$, $\forall i \in C$.

Finally, we point out that the above model can also be easily modified to a more general location-routing variant of the MDSDVRP. To this end, binary decision variables indicating active depots need to be added together with appropriate linking constraints ensuring that the degree of each unused depot is zero. Capacity constraints for each depot (as, e.g., proposed by Belenguer et al. (2011)) may need to be considered as well as depot-opening costs in the objective function. If depots have capacity limits, single-customer route variables between each customer and depot are required instead of those used in formulation (1). All inequalities proposed in the following sections remain, however, valid also for this case and one could also use our algorithmic framework by implementing these small adaptations. We do not consider such a location-routing problem variant in our study as it would not induce additional scientific contributions aside from additional computational results.

3 Single-customer route constraints

This section discusses valid inequalities that exploit single-customer route variables $\mathbf{y}$. Recall, that the role of these variables is to separate the modeling of single- from multi-customer routes. Formulation (1) allows to represent a single-customer route from depot $d \in D$ to customer $i \in C$ by single-customer route variables (i.e., $\bar{y}_i = 1$) or, alternatively, via edge variables (i.e., $\bar{x}_{di} = 2$). These undesired symmetries are excluded via *symmetry breaking constraints* (3) and (4).

$$x(\delta_D(j)) + y_j \leq z_j \qquad \forall j \in C \tag{3}$$

$$y_j + x_{ij} \leq \left\lceil \frac{q_j}{Q} \right\rceil \qquad \forall j \in C, \forall \{i,j\} \in E_C \tag{4}$$

Constraints (3) ensure that variable x_{dj}, $\{d,j\} \in E_D$, cannot be used for a single-customer route as its value would be equal to two while each single-customer route contributes only one to variable z_j via degree constraints (1b). Inequalities (4) ensure that customer $j \in C$ cannot be part of a multi-customer route if its demand can be covered by single-customer routes visiting j.

Next, we introduce three sets of inequalities that focus on customers with demands (almost) equal to the vehicle capacity.

$$y_i + z_i \geq 2 \qquad \forall i \in C : q_i = Q \tag{5}$$

$$2z_i - x(\delta(i,D)) \geq 2 \qquad \forall i \in C : q_i = Q \tag{6}$$

$$y_i + x(\delta(i,D)) + z_i \geq 2 \qquad \forall i \in C : q_i = Q - 1 \tag{7}$$

Inequalities (5) are based on the observation that each customer i with demand $q_i = Q$ is either served by a single-customer route or needs to be visited at least two times. Inequalities (6) hold since each customer $i \in C$ such that $q_i = Q$ must either be served by one single-customer route or via two routes containing multiple customers. Inequalities (7) make sure that each customer whose demand is equal to $Q - 1$ is either part of a route with at most two nodes (which implies that it is adjacent to a depot) or visited by at least two vehicles.

4 Capacity constraints

As indicated above, no set of linear constraints ensuring the capacity limits vehicles is known in natural space of edge (or arc) variables for routing problems in which demand nodes (or demand arcs) can be visited multiple times, see, e.g., Archetti et al. (2014); Ozbaygin et al. (2018); Belenguer and Benavent (2003); Bode and Irnich (2012). One line of research, therefore, uses extended formulations in which variables represent, e.g., vehicle routes (Archetti et al., 2011) or pairs of subsequently traversed edges (Munari and Savelsbergh, 2019). These variables allow the derivation of inequalities that enforce the vehicle capacities. An alternative that has led to state-of-the-art results for SDVRP variants and which is also pursued in this work is to use or adapt (rounded) capacity constraints and then eliminate infeasible solutions that are not cut-off by them. Most papers focusing on the latter technique use variants of the rounded capacity constraints (8).

$$x(E(S)) \leq z(S) - \left\lceil \frac{q(S)}{Q} \right\rceil \qquad \forall S \subseteq C. \tag{8}$$

Their validity and the fact that these constraints ensure that the number of vehicles entering set S is at least $\left\lceil \frac{q(S)}{Q} \right\rceil$ can be seen more easily in their cut form $x(\delta(S)) + 2y(S) \geq 2 \left\lceil \frac{q(S)}{Q} \right\rceil$, $\forall S \subseteq C$, introduced by Dror et al. (1994). The latter is obtained from (8) via the degree constraints (1b).

Next, we introduce a new set of capacity constraints (9) denoted by yCAPs that exploit single-customer route variables. They are based on the observation that the residual capacity of a vehicle performing a single-customer route cannot be used to satisfy further demands.

$$x(E(S)) \leq z(S) - \frac{q(S)}{Q} - \sum_{i \in S: q_i < Q} \left(1 - \frac{q_i}{Q}\right) y_i \qquad \forall S \subseteq C \tag{9}$$

Theorem 1. *Capacity constraints* (9) *are valid for the MDSDVRP.*

The missing rounding on the right-hand side of (9) indicates that yCAPs do not theoretically dominate the rounded capacity constraints (8). Indeed, the computational results discussed in Section 7 show that none of these two sets of inequalities dominate the other, i.e., that the bounds obtained from the corresponding linear programming relaxations are incomparable.

Corollary 1. *Rounded capacity constraints* (8) *and yCAPs* (9) *do not imply each other.*

Theorem 2 shows the aforementioned fact that even the combination of rounded capacity constraints (8) and yCAPs (9) do not guarantee that the capacity limits of all vehicles are met.

Theorem 2. *Consider formulation* (1) *when generic capacity constraints* (1f) *are replaced by rounded capacity constraints* (8) *and yCAPs* (9)*. There exist solutions to this formulation that are infeasible for the MDSDVRP even if the depot-consistency constraints are met.*

We conclude this section by pointing out that another generalization of rounded capacity constraints (the so-called wCAPs) exploiting single-customer route variables has been proposed by Belenguer et al. (2011) for the CLRP. These inequalities can be easily adapted to the MDSDVRP by considering the number of node visits $z(S)$ in a set $S \subseteq C$ instead of using its cardinality $|S|$ on their right-hand side. As the lower bounds obtained by using the resulting inequalities in preliminary computational experiments were equivalent to those from the rounded capacity constraints (8), we do not consider this adaptation of the wCAPs to the MDSDVRP in the following.

5 Depot-consistency constraints

Ensuring that each vehicle returns to the depot it started from is the main additional challenge of multi-depot routing problems when compared to their single-depot counterparts. Proposing so-called *chain barring constraints*, Laporte et al. (1984) were among the first to propose and study a family of inequalities ensuring this property in the natural space of undirected edge-variables. These inequalities eliminate paths between two depots by ensuring that the end nodes of any path cannot be connected to two different depots. Laporte et al. (1988), Belenguer et al. (2011) and Benavent and Martínez (2013) generalized these *path-elimination (PE)* constraints. These constraints are the basis for the new PE based depot-consistency constraints proposed in Section 5.1.

Bektaş et al. (2017) introduced a new set of cut-like inequalities for asymmetric multi-depot routing problems and showed that they eliminate paths between different depots when each customer is visited (at most) once. As these constraints focus on enforcing that each vehicle returns to its original depot rather than explicitly excluding infeasible paths, they have been labeled as *path-forcing (PF) constraints*. In Section 5.2, we will build on this idea and propose a new family of cut-like inequalities for undirected multi-depot routing problem with customer revisits.

A new set of PE constraints for asymmetric multi-depot routing problems has been recently proposed by uit het Broek et al. (2021). These inequalities are based on D_k^+ and D_k^- constraints for the asymmetric traveling salesperson problem (Grötschel and Padberg, 1985) and are inherently asymmetric which is why we do not consider them in the remainder of this article.

5.1 Path-elimination based constraints

Belenguer et al. (2011) introduced PE constraints for multi-depot routing problems without node revisits. These constraints (and variants known from the literature) are, however, not valid for the MDSDVRP since they eliminate solutions that contain paths connecting different depots. As discussed earlier, some of these paths are feasible when nodes can be visited multiple times. The inequalities by Belenguer et al. (2011) consider two dedicated customers i and j, partition the set of depots into two sets $I \subseteq D$ and $D \setminus I$, and then consider the set of edges $\delta(i, I) = \{\{i, u\} \in E \mid u \in I\}$ and $\delta(j, D \setminus I) = \{\{j, u\} \in E \mid u \in D \setminus I\}$. We will use the term *leg* to refer to such a set of edges between one customer and a set of depots. Next we propose a family of *multi-leg PE constraints* (10). They are valid for multi-depot routing problems in which customers can be visited multiple times and generalize the inequalities from Belenguer et al. (2011) in the following two ways: (i) the right-hand side is changed to a term depending on the total number visits of the nodes in S; and (ii) more than two legs are considered.

$$\sum_{j=1}^{k} x(\delta(i_j, I_j)) + y(S) + x(E(S)) \leq z(S) \qquad \forall S \subseteq C, i_j \in S, I_j \subseteq D, i_j \neq i_p$$

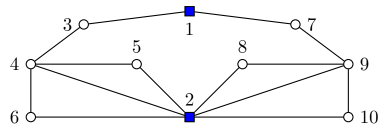

Figure 3: An infeasible candidate solution that does not violate multi-leg PE constraints (10).

$$I_j \cap I_p = \emptyset, 1 \leq j \neq p \leq k \tag{10}$$

The first generalization ensures validity of inequalities (10) in the context of multiple visits, see Theorem 3. Theorem 4 shows that the second generalization allows to cut-off infeasible solutions that do not violate the subset of inequalities considering only two legs.

Theorem 3. *Multi-leg PE constraints* (10) *are valid for the MDSDVRP.*

Theorem 4. *Consider formulation* (1) *when generic depot-consistency constraints* (1g) *are replaced by the subset of multi-leg PE constraints* (1g) *for* $k = 2$. *There exist solutions to this formulation that violate at least one multi-leg PE constraint* (1g) *for* $k \geq 3$.

The next two results show that multi-leg PE constraints (10) do not ensure that each subgraph satisfying them can be decomposed into a set of routes that each start and end at the same depot.

Theorem 5. *Consider formulation* (1) *when generic depot-consistency constraints are replaced by multi-leg PE constraints* (10). *There exist solutions to this formulation that are infeasible for the MDSDVRP even if the capacity restrictions of all vehicles are met.*

Corollary 2. *Formulation* (1) *with multi-leg PE constraints* (10) *in place of* (1g) *models only a relaxation of the MDSDVRP even if the capacity restrictions of all vehicles are met.*

5.2 Path-forcing based constraints

Bektaş et al. (2017) introduced PF constraints for asymmetric multi-depot routing problems that ensure that each vehicle returns to its initial depot without visiting other depots. The fact that this property is not ensured by multi-leg PE constraints (cf., Figure 3 and the discussion of this example in the proof of Theorem 5) motivated the inequalities proposed in this section. Though these PF constraints are valid for multi-depot routing problems with split-deliveries, they cannot be used without introducing new sets of variables indicating how often each edge is traversed in which direction. Doing so would add symmetries to our formulations and thus likely deteriorate the computational performance of our solution algorithms. Instead, we propose a new set of undirected cut-like inequalities that transfer the ideas of the directed PF constraints by Bektaş et al. (2017) to the space of undirected edge variables. *PF constraints* (11) state that if edge $\{d, i\} \in E_D$ is in a solution, then any cut separating $d \in D$ and $i \in C$ must contain an edge different from $\{d, i\}$.

$$2x_{di} \leq x(\delta(S, (C \cup \{d\}) \setminus S)) \qquad \forall d \in D,\ \forall S \subseteq C,\ \forall i \in S \tag{11}$$

The validity of PF constraints (11) will be shown when we introduce the more general *multi-leg PF constraints*. Observe that the candidate solution from Figure 3 does not contain a route starting and ending at depot 1 that does not visit depot 2 and therefore violates several PF constraints (11), e.g., for $d = 1$, $i = 3$, and $S = \{3, 4, 5, 6\}$. As shown in the proof of Theorem 5, the candidate solution satisfies all multi-leg PE constraints (10). These observations imply the following corollary.

Corollary 3. *Consider formulation* (1) *with generic depot-consistency constraints replaced by multi-leg PE constraints* (10). *There exist solutions to this formulation that violate at least one PF constraint* (11).

The discussion related to multi-leg PE constraints suggests to consider multiple depot edges incident to different depots and to account for the fact that these edges must be part of different vehicle routes in case they are used. Indeed, *multi-leg PF constraints* (12) generalize inequalities (11) by considering several legs linked to different customers from S on their left-hand side.

$$2\sum_{j=1}^{k} x(i_j, I_j) \leq x(\delta(S, (C \cup D') \setminus S)) \qquad \forall D' \subseteq D,\ \forall S \subseteq C,\ i_j \in S,\ D' = \cup_{j=1}^{k} I_j,$$
$$I_j \cap I_l = \emptyset, 1 \leq j \neq l \leq k \tag{12}$$

These inequalities state that each vehicle leaving some depot $d \in D$ via an edge $\{d, i\}$ considered on their left-hand side to first visit a customer $i \in S$ must leave S to visit a customer in $C \setminus S$ or to return to depot in a set D' which must

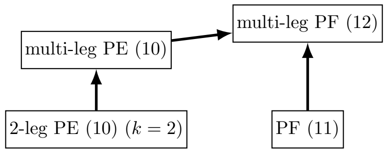

Figure 4: Relations between depot-consistency constraints. An arrow indicates that the set of inequalites at the head dominates the one at the tail.

contain $d \in D$. Furthermore, it cannot use the initial edge $\{d, i\}$ to leave set S. Theorem 6 shows that multi-leg PF constraints are valid for the MDSDVRP. Since multi-leg PF constraints (12) contain inequalities (11) as a special case when $I_j = \{d\}$, $i_j \in S$, and $D' = \{d\}$, validity of the latter inequalities also follows from Theorem 6. Theorems 7 and 8 clarify the relation of multi-leg PF constraints (12) to their simpler versions (11) and to multi-leg PE constraints (10), respectively. Together, Theorem 8 and Corollary 3 imply Corollary 4.

Theorem 6. *Multi-leg PF constraints* (12) *are valid for the MDSDVRP.*

Theorem 7. *Consider formulation* (1) *when generic depot-consistency constraints* (1g) *are replaced by PF constraints* (11)*. There exist solutions to this formulation that violate at least one multi-leg PF constraint* (12)*.*

Theorem 8. *Multi-leg PF constraints* (12) *imply multi-leg PE constraints* (10)*.*

Corollary 4. *Consider formulation* (1) *with generic depot-consistency constraints replaced by multi-leg PE constraints* (10)*. There exist solutions to this formulation that violate at least one multi-leg PF constraint* (12)*.*

Theorem 9 and Corollary 5 reveal that multi-leg PF constraints (12) are not sufficient to obtain a valid model for the MDSDVRP. Figure 4 summarizes established relations between depot-consistency inequalities.

Theorem 9. *Consider formulation* (1) *with generic depot-consistency constraints replaced by multi-leg PF constraints* (12)*. There exist solutions to this formulation that are infeasible for the MDSDVRP and which do not violate the capacity restrictions of any vehicle.*

Corollary 5. *Formulation* (1) *with multi-leg PF constraints* (12) *in place of* (1g) *models only a relaxation of the MDSDVRP even if the capacity restrictions of all vehicles are met.*

6 Algorithm

This section describes the developed B&C framework based on Gurobi 9.1.2. In addition to the separation of valid inequalities, a primal heuristic, and the feasibility check for integer solutions are detailed. The algorithm solves the LP relaxation of (1) via a dedicated cutting plane algorithm that enables full control over added cuts and which uses Gurobi only as LP solver. The resulting set of valid inequalities is then used to initialize Gurobi's branch-and-cut algorithm which potentially adds further inequalities via callback routines. Constraints (1b)–(1d) and symmetry breaking constraints (3) are added to our initial model which also includes the following constraints that eliminate some single-customer routes and impose a bound on the number of splits: (i) Since the vehicle limit K induces a lower bound $Q_{\min} = \max\{0, \sum_{i \in C} q_i - (K-1) \cdot Q\}$ on the demand each vehicle has to serve, we set $y_i = 0$ for every customer $i \in C$ with $q_i < Q_{\min}$; (ii) following Archetti et al. (2014) who showed (for the SDVRP) that there exists an optimal solution in which the number of splits $z(C) - |C|$ is smaller than the number $z(D)$ of routes, we add $z(C) - z(D) \leq |C| - 1$.

6.1 Heuristic

We propose a heuristic for the MDSDVRP (and all considered variants) which we use to initialize the B&C algorithm. It represents routes as customer sequences and therefore avoids depot decisions. Instead, a best depot is selected when evaluating the cost of route changes.

The first solution is constructed by the savings heuristic of Clarke and Wright (1964) that starts with single-customer routes and iteratively and feasibly merges two (potentially reversed) routes that lead to a maximal cost reduction, until no further improvement exists. In this phase no customer splits are considered. The resulting solution is improved by large-neighborhood search (LNS), see, e.g., Ropke and Pisinger (2006). The LNS which iteratively diversifies and intensifies the incumbent solution stops in case of 5 000 consecutive non-improving iterations (100 for the MDTSP without split deliveries) or when its runtime reaches 10% of the overall time limit. In the diversification phase, ℓ randomly selected customers are removed from all routes and served instead via single-customer routes. Parameter ℓ is initialized to one, increased after 1 000 iterations without improvement, and reset to one in case a new best solution is found. In the improvement phase, the customers are considered one by one in random order, removed from all routes, and inserted again in a best possible way. This phase stops when no improvement occurred. When split deliveries are allowed, optimally inserting a customer requires the identification of a set of routes whose joint free capacity is at least the customer's demand such that the total insertion costs are minimal. Such a set is identified via dynamic programming. After this improvement phase, we check whether merging routes, as done in the initial solution construction, leads to further cost reductions.

At the end of each iteration, a post-processing step is performed to ensure that the obtained solution is feasible for our MIP formulation. We reduce the number of routes in case the resulting solution exceeds the vehicle limit K. For this, as long as the vehicle limit is violated, we select a route with minimal total demand served, remove all its customers from the solution, and re-insert them as in the improvement phase. Finally, the improvement phase described above is applied again. Then, we remove potentially existing, not necessary split deliveries of customers served by single-customer routes to satisfy symmetry breaking constraints (3)–(4), and eliminate customer cycles using a procedure from Archetti et al. (2006b) to satisfy SECs (1e).

To each incumbent solution that is found by the MIP solver we apply a single improvement step (without diversification phase) followed by above mentioned post-processing and return the resulting solution to the solver in case of a cost reduction.

6.2 Separation of inequalities

Inequalities are considered in the order described below and we only add cuts whose violation is at least 0.05. At most 100 cuts are added in each iteration for fractional solutions. For integer solutions, the separation stops immediately when violated inequalities of some type are found. To avoid convergence problems, the cutting plane algorithm to compute the initial LP relaxation stops if the cumulative increase of the dual bound of 10 consecutive iterations is smaller than one. Notation $\bar{E} = \{e \in E \mid \bar{x}_e > 0\}$, $\bar{E}_D = \bar{E} \cap E_D$, and $\bar{E}_C = \bar{E} \cap E_C$ will be used below.

Symmetry breaking constraints (4) are separated by enumeration over all customers $i \in C$ such that $\bar{y}_i \geq 0.05$.

Subtour elimination constraints (1e) are separated using depth-first search for integer solutions and the procedure described in Padberg and Grötschel (1985), page 327, for fractional solutions. All customer subsets found are sorted by non-decreasing size and only SECs whose customer sets are disjoint to those already added in the current iteration are added.

***y*CAPs** (9) are separated only for fractional solutions using a minimum cut based procedure similar to the one by Blasum and Hochstättler (2000) for fractional capacity cuts for VRPs. Details can be found in the electronic companion. For each identified set $S \subseteq C$ violating a yCAP, we also check whether it violates a rounded capacity cut (8) in which case we add this constraint, too.

Rounded capacity cuts (8) are first separated heuristically by i) checking sets S of violated yCAPs and ii) applying the heuristics in the CVRPSEP package by Lysgaard et al. (2004). In this second case, and since the CVRPSEP routines are designed for the CVRP with a single depot, we merge all depots (and incident edges) to a single depot and convert single-customer routes to corresponding depot edges. Since CVRPSEP does not consider multiple node visits some of the identified sets S may not correspond to violated inequalities. These sets are simply discarded. An exact, MIP-based separation routine (detailed in the electronic companion) is applied in the dedicated cutting plane algorithm, in the root node of the branch-and-cut tree, and for all integer solutions in case no cuts are found by the CVRPSEP heuristics.

Path-elimination constraints (10) are separated only for integer solutions since preliminary experiments showed that the dual bound increase obtained from separating them for fractional solutions is too small to justify the separation time. Even though $|S|$ is replaced by $z(S)$ on their right-hand side, PE constraints (10) with two legs can be separated in polynomial time using the method from Belenguer et al. (2011) which computes a minimum cut between each pair of customers (that represent the leg endpoints in set S). If no PE constraints with two legs can be found, the exact separation detailed in B.3 is used.

Path-forcing constraints (12) are separated in two steps. First, we identify violated single-leg PF constraints (11) using a minimum-cut approach. For each depot edge $\{d, i\} \in \bar{E}_D$, we compute a minimum-cut between customer i and depot d in the support graph induced by $\bar{E} \cap E(C \cup \{d\})$. Cutset $S \subset C$ induces a violated PF inequality (11) if the cut value is less than $2\bar{x}_{di}$. For integer solutions, the exact separation of multi-leg PF constraints (12), detailed in B.3, is applied if no single-leg inequalities are violated.

Lifting of depot-consistency constraints. As detailed in the proof of Theorem 8, multi-leg PF constraints can be equivalently written in subtour form as

$$\sum_{j=1}^{k} x(\delta(i_j, I_j)) + 0.5x(\delta(S, D \setminus D')) + x(E(S)) + y(S) \leq z(S)$$

for $S \subseteq C$, $i_j \in S$, $I_j \subseteq D$, $i_j \neq i_p$, $I_j \cap I_p = \emptyset$, $1 \leq j \neq p \leq k$. This makes it easy to see that they dominate all depot-consistency constraints discussed in this article and gives rise to the following lifting procedure that is applied to all violated (multi-leg) PE and PF constraints. Keeping customer set S fixed, for each depot $d \in D$ we determine its incident edge with largest solution value, i.e., $\{d, i^*\} = \arg\max_{\{d,i\} \in E_D : i \in S} \bar{x}_{di}$. If $\bar{x}_{di^*} > 0.5\bar{x}(\delta(d, S))$, then edge $\{d, i^*\}$ is added to the corresponding leg (i.e., selected depot edges incident to node i^*), otherwise depot d is not part of D' and thus involved in the term with coefficient 0.5.

6.3 Feasibility check for integer solutions

As already discussed before, a feasible solution to relaxation (1) enhanced with all inequalities discussed in this paper, might still be infeasible for the MDSDVRP in the sense that it cannot be decomposed into a set of routes such that each vehicle starts and ends at the same depot and its capacity is not exceeded. We identify such solutions using a set-cover formulation proposed by Bianchessi and Irnich (2019) that relies on enumerating all possible routes and that has shown to be a simple and fast method, see also Archetti et al. (2014); Bianchessi et al. (2019).

Before applying the formulation, we perform the following steps which help to reduce the number of times it needs to be solved and its size. We first observe that each solution in which each customer is visited exactly once is feasible (otherwise it would have been cut-off by a depot-consistency or capacity inequality) and we can therefore skip the feasibility check in this case. Next, single-customer routes are removed and the demand q_i of each customer $i \in C$ visited by such a route is reduced to $\max\{0, q_i - \bar{y}_i Q\}$. The number of used vehicles $\bar{K} = \bar{z}(D)$ is then used to compute a lower bound $\bar{Q}_{\min} = \max\{0, q(C) - (\bar{K} - 1) \cdot Q\}$ of the demand each vehicle must serve. Next, we generate all routes $\bar{R}$ contained in $(V, \bar{E})$ that i) visit each customer at most once, ii) start and end at the same depot, and iii) can potentially satisfy capacity restrictions. For the latter, we calculate the required and optional demands of route r. The required demand $\bar{D}^{\mathrm{r}}$ is the total demand of customers from that route that are visited only once in the solution. The optional demand $\bar{D}^{\mathrm{o}}$ is the sum of demands of all visited split customers. A route can satisfy the capacity restrictions only if $\bar{D}^{\mathrm{r}} + \bar{D}^{\mathrm{o}} \geq \bar{Q}_{\min}$ and $\bar{D}^{\mathrm{r}} \leq Q$. If a customer is not included in any route, the solution is infeasible.

Formulation (13) proposed by Bianchessi and Irnich (2019) is used to find a decomposition of the current solution into a set of routes from $\bar{R}$. It includes binary variable $\lambda_r \in \{0, 1\}$ for each route $r \in \bar{R}$, and integer variables $\delta_{ri} \geq 0$ indicating the demand at split customer $i \in \bar{C}^{\mathrm{s}} = \{i \in C \mid \bar{z}_i \geq 2\}$ covered by route $r \in \bar{R}$. The value c_r denotes the total cost of route $r \in \bar{R}$. Let $C_r \subseteq C$ be the set of customers visited by route $r \in \bar{R}$, and $E_r \subseteq \bar{E}_C$ be the set of customer edges used by route r.

$$\min \quad \sum_{r \in \bar{R}} c_r \lambda_r \tag{13a}$$

$$\text{s.t.} \quad \sum_{r \in \bar{R} : i \in C_r} \lambda_r \geq \bar{z}_i \qquad \forall i \in C \tag{13b}$$

$$\sum_{r \in \bar{R} : i \in C_r} \delta_{ri} = q_i \qquad \forall i \in \bar{C}^{\mathrm{s}} \tag{13c}$$

$$\sum_{i \in C_r \cap \bar{C}^{\mathrm{s}}} \delta_{ri} + \sum_{i \in C_r \setminus \bar{C}^{\mathrm{s}}} q_i \lambda_r \leq Q \lambda_r \qquad \forall r \in \bar{R} \tag{13d}$$

$$\sum_{r \in \bar{R} : e \in E_r} \lambda_r \leq 1 \qquad \forall e \in \bar{E}_C \tag{13e}$$

$$\sum_{r \in \bar{R} : |C^{\mathrm{s}} \cap C_r| = 1} \lambda_r \geq 2 \tag{13f}$$

Constraints (13b) ensure that all customers are visited at least as often as in the solution, (13c) guarantee that demands of all split customers are satisfied, (13d) make sure that each route respects the vehicle capacities, (13e) assure that each customer edge is traversed at most once, and finally, (13f) guarantee that there are at least two routes with a single split customer, see Property 6. If the MIP is infeasible or the resulting route subset does not correspond to the current solution, we add an associated feasibility cut (2). In the latter case, we check whether the route subset leads to a new incumbent solution in which case we hand it over to the branch-and-cut framework.

7 Computational Experiments

In this section we discuss the results obtained from applying our algorithm to data sets for the MDSDVRP, the MDTSP, and the SDVRP. After describing the benchmark instances used and mentioning the solution methods from the literature to which we compare our results to (Section 7.1), we proceed with analyzing the impact of the inequalities proposed in this work on LP-relaxation bounds and on the performance of the B&C algorithm (Section 7.2). Section 7.3 compares the best performing B&C variant to state-of-the-art methods and results on the MDSDVRP, the MDTSP, and the SDVRP. The electronic companion also contains an assessment of the solution quality of the proposed heuristic, see Appendix F.

We used Gurobi 9.1.2 with its default parameter settings except for the following: We set the number of threads to one (`Threads`= 1), the focus to optimality (`MIPFocus`= 2), the MIP gap tolerances to zero (`MIPGap`=`MIPGapAbs`= 0.0), and the time ratio for MIP heuristics to 10% (`Heuristics`= 0.1). A time limit of two hours and a memory limit of 8 GB is used. The only exception to this setting are the experiments in Section 7.2.1 used to compute LP-relaxation gaps.

7.1 Instances and comparison

For the MDSDVRP, we consider the set of 42 instances from Gulczynski (2010) that contains 12 (`sq`) instances created with a focus on allowing to visually estimate solution quality and 30 (`mdsd`) instances that are based on 10 multi-depot VRP instances. Each of the latter is the base of three MDSDVRP instances that differ by the range of customer demands $D_{\mathrm{R}} \in \{[.1, .9], [.3, .7], [.7, .9]\}$ relative to the vehicle capacity, e.g., $D_{\mathrm{R}} = [.3., .7]$ indicates $0.3Q \leq q_i \leq 0.7Q$ for all $i \in C$. Results on those instances which contain up to 249 customers and 5 depots are compared to those of the heuristic by Gulczynski et al. (2011). To facilitate an analysis of LP-gaps and different B&C configurations, we also created a set of 27 smaller instances (denoted to as `mdsd40`) with $|C| = 40$, $|D| \in \{2, 4, 6\}$, and $D_{\mathrm{R}} \in \{[.1, .9], [.3, .7], [.7, .9]\}$. These are created from the corresponding instances from set `mdsd` by considering only the first 40 customers and only the first $|D|$ depots.

For the MDTSP, we consider the set of instances with symmetric costs created by Bektaş et al. (2017) (`BGS`) and the largest instances from Benavent and Martínez (2013) (`BM`). The latter ones were originally proposed for the location routing problem (see, `http://prodhonc.free.fr/`). The instances from both sets contain up to 300 customers and 60 depots. We compare the performance of our B&C to the best B&C results from these two papers.

We also run our algorithm for the SDVRP data sets used in Archetti et al. (2014) and Munari and Savelsbergh (2019). These 352 instances are grouped into four common SDVRP scenarios treated in the literature (limited / unlimited fleet size; rounded / non-rounded costs). The values of best known lower and upper bounds are those reported in Archetti et al. (2011); Munari and Savelsbergh (2019), and are therefore based on the results from Hernández-Pérez and Salazar-González (2019); Ozbaygin et al. (2018); Chen et al. (2017); Silva et al. (2015); Archetti et al. (2011, 2014); Belenguer et al. (2000); Munari and Savelsbergh (2019). Overall, 276 of these instances have been tackled in these papers. Lower and upper bounds are reported for 276 and 267 of them, respectively.

7.2 Analysis of capacity and depot-consistency constraints

In this section, we analyze the impact and effectiveness of capacity and depot-consistency constraints derived in Sections 4 and 5, respectively. To gain insights we first discuss in Section 7.2.1 results from solving the LP relaxation of different variants of formulation (1) using a dedicated cutting plane algorithm and exact separation routines for all inequalities included. Since the exact solution of LP relaxations is time consuming, we restrict this discussion to instances from set `mdsd40` for the MDSDVRP and the aforementioned MDTSP instances. As will be detailed below, this analysis shows (i) the relevance of yCAPs (9) as they significantly strengthen the dual bounds obtained from rounded capacity constraints (8); (ii) the positive impact of the depot-consistency constraints on dual bounds; (iii) that there exist several cases in which multi-leg PE (10) and multi-leg PF constraints (12) are (significantly) stronger than their simpler variants (i.e., two-leg PE and single-leg PF constraints (11)); and (iv) that the dual bounds obtained by multi-leg PE and multi-leg PF constraints are (almost) identical for the MDSDVRP while the bounds obtained by the latter are significantly tighter in case of the MDTSP.

Section 7.2.2 then focuses on the impact of selecting different sets of inequalities on the performance of the B&C algorithm and on the number of cases in which infeasible candidate solutions cannot be cut-off by them. This section uses the more sophisticated (heuristic) separation strategies detailed in Section 6.2. One main result of this analysis is the observation that the combination of all inequalities developed in the previous sections yields the overall best-performing algorithm.

7.2.1 Impact on LP-relaxation

Figure 5 shows LP gaps (computed as the difference between the LP-relaxation value and the optimal solution value relative to the latter) in percent for different MDSDVRP instances (from set `mdsd40`), numbers of depots, and demand

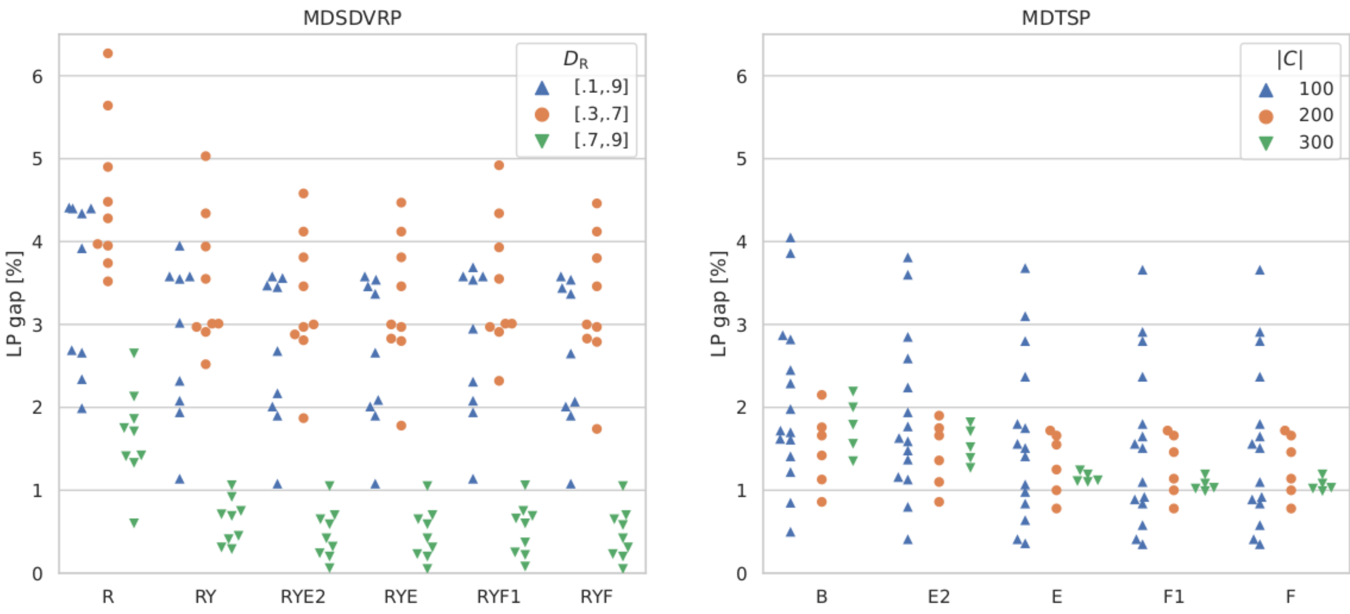

Figure 5: LP-relaxation gaps of cutting plane algorithms for MDSDVRP (set `mdsd40`) and MDTSP instances.

intervals. To analyze the impact of yCAPs, we first compare variants `R` (considering only rounded capacity-cuts (8)) and `RY` (additionally considering yCAPs (9)) which both do not include any depot-consistency constraints. Considering our base variant `R`, we observe a strong relation between gaps and customer demands. With one exception (instance mdsd40-6 for $|D| = 6$) the largest and smallest gaps are observed for demand ranges $[.3, .7]$ and $[.7, .9]$, respectively. No clear dependency of LP-gaps on the number of depots can be observed. We observe that yCAPs improve the LP-relaxation values on all considered instances and settings. Their effect is significant for all demand ranges and the largest one for the high-demand case. The relative amount (rounded to one decimal place) of the gap of variant `R` that is closed by yCAPs lies in the intervals $[31.6\%, 79.4\%]$, $[10.4\%, 42.7\%]$ and $[17.1\%, 32.8\%]$ for demand ranges $[.7, .9]$, $[.1, .9]$, and $[.3, .7]$, respectively.

Next, we focus on the impact of different depot-consistency constraints on LP gaps. For the MDSDVRP, this analysis uses the results shown in Figure 5 and Table 2 with `RY` as baseline and considers variants also including two-leg PE (`RYE2`), multi-leg PE (`RYE`), (single-leg) path-forcing (`RYF1`), or multi-leg PF (`RYF`) constraints. We observe that two-leg PE constraints further reduce the already relatively small gaps of `RY` in 25 out of 27 cases. On average 13.4% of the gap remaining after considering all capacity constraints is closed by them and the maximum gap reduction is 80.6% (instance mdsd40-1, $|D| = 6$, $D_R = [.7, .9]$). The impact of also considering multiple legs in PE constraints seems minor on the instances and settings considered in Table 2. In the majority of settings (15 cases) the gap of `RYE` is identical to the one of `RYE2` and only relatively small differences can be observed in most of the other cases. We also conclude that two-leg PEs (`RYE2`) seem preferable to (single-leg) PF constraints (`RYF1`) on the considered instances and settings as the gaps of the latter are always at least as high as the one of the former. The gaps resulting from including multi-leg PF constraints (`RYF`) are almost always identical to those obtained by multi-leg PE constraints (`RYE`) and only minor improvements can be observed in 7 out of 27 cases.

A different picture can be observed for the uncapacitated case of the MDTSP, see Figure 5 and Table 5. Here, we compare the base case (`B`) not including any depot-consistency constraints to variants including different sets of them. We observe that considering multiple legs in PE constraints has a significant impact in this case. The LP gaps of `E` are (often significantly) smaller than those of `E2` in 24/26 cases. Interestingly and in contrast to the results for the MDSDVRP, the bounds obtained by including single-leg PF constraints (`F1`) are at least as small as those when including multi-leg PEs (`E`) and better in the majority of cases (16/26). Finally, we also observe that using multi-leg PF constraints (`F`) does not lead to any further gap reductions (compared to variant `F1`) for the MDTSP at least for the instances considered in this set of experiments.

7.2.2 Impact on branch-and-cut

This section compares variants of the B&C algorithm including different sets of inequalities (separated as detailed in Section 6.2) using instance set `mdsd40` for the MDSDVRP. Besides the identification of a most-promising variant we also analyze how many candidate solutions violating capacity or depot-consistency restrictions can only be identified by the feasibility check. To facilitate the latter, we slightly modify the feasibility check for these experiments. We generate all routes contained in the support graph of a candidate solution that visit a customer at most once and start and end at the same depot, but do not take into account vehicle capacities in this step. For each infeasible

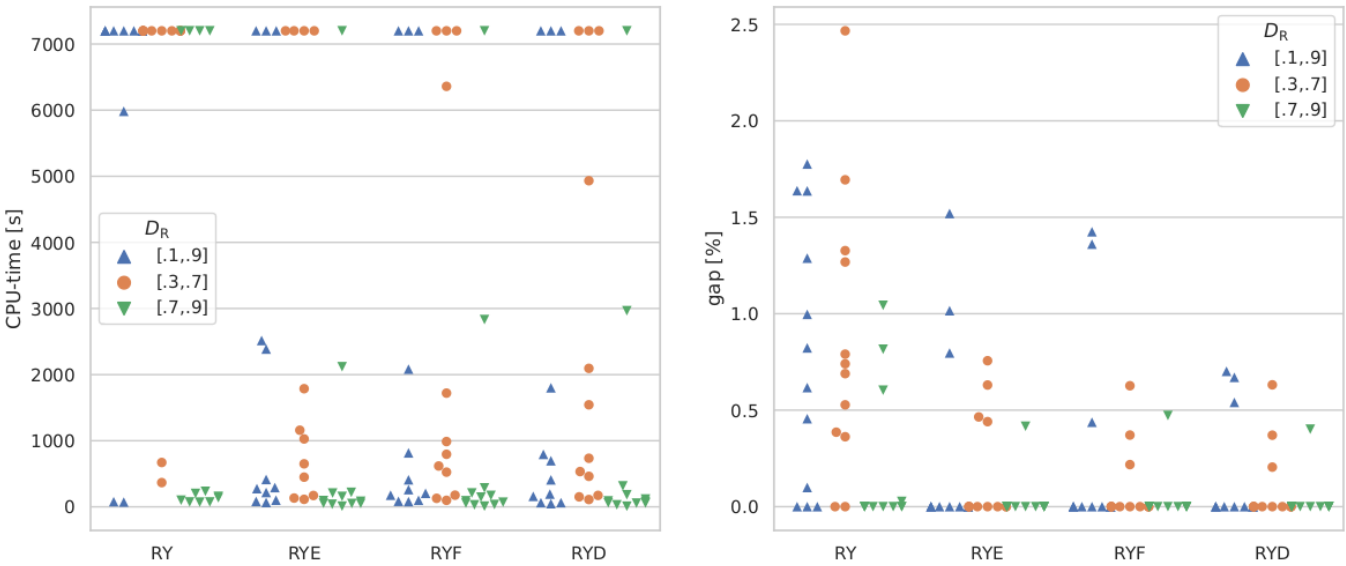

Figure 6: Solution times and optimality gaps of B&C variants for the MDSDVRP on instance set `mdsd40`.

candidate solution, we re-solve MIP (13) without constraints (13c) and (13d) that ensure the capacity restrictions. If this modified MIP is infeasible, the solution violates depot-consistency requirements, otherwise it can be feasibly decomposed into a set of routes that each start and end at the same depot but violates capacity restrictions.

Figure 6 shows CPU-times and optimality gaps after two hours of four B&C variants that all include symmetry breaking, subtour elimination, and rounded capacity cuts constraints. Their abbreviations indicate whether they also consider yCAPs (`Y`), (multi-leg) PE constraints (`E`), (multi-leg) PF constraints (`F`), or all depot-consistency constraints (`D`). The lifting described in Section 6.2 is applied for all variants including depot-consistency constraints.

In line with the observation made w.r.t. LP-relaxation gaps, the results shown in Figure 6 clearly indicate a dependency between the difficulty of instances and the demand range. Instances with medium demands ($D_{\mathrm{R}} = [.3, .7]$) appear to be the most difficult ones while those with high demands ($D_{\mathrm{R}} = [.7, .9]$) are the easiest ones. Comparing `RY` to the other variants, we also observe a significant positive impact of including depot-consistency constraints both concerning the number of instances solved as well as the remaining gaps. The detailed results in Table 3 show that all 13 (out of 36) instances solved by `RY` are also solved by the other three variants which solve 15 (`RYE`) or 16 (`RYF`, `RYD`) additional instances. With a single exception (mdsd40-9, $|D| = 2$, $D_{\mathrm{R}} = [.1, .9]$), also the optimality gaps of unsolved instances decrease when including any subset of depot-consistency inequalities. The performance of the three B&C variants with depot-consistency constraints is quite similar. Slight advantages can be observed, however, for variant `RYD` considering all proposed sets of inequalities which performs best (i.e., either has a smallest runtime or gap) on 22 instances.

Next, we turn our attention to candidate solutions that violate capacity or depot-consistency restrictions whose infeasibilities can only be identified by the feasibility check. Figure 7 shows numbers of such solutions per instance and reason of infeasibility. Since the significance of these numbers is questionable for instances not solved optimally, Figure 7 also indicates instances solved by all variants (`A`), only by variants `RYE`, `RYF`, and `RYD` considering at least one family of depot-consistency constraints (`D`), or by none of the considered variants (`N`). The only case (mdsd40-2, $|D| = 6$, $D_{\mathrm{R}} = [.3, .7]$) that does not fit into this classification is included in `D` as it is solved by two variants including depot-consistency constraints (`RYF`, `RYD`). We observe that the fact that neither capacity nor depot-consistency constraints are sufficient has not only theoretical relevance but can be observed frequently. The results also indicate a strong correlation between numbers of infeasible candidate solutions that can only be cut-off by inequalities (2) and the difficulty of an instance. They also show the relevance of including (multi-leg) PF or all depot-consistency constraints as they are able to significantly reduce the number of solutions whose infeasibility is only identified during the feasibility check, at least on the instances considered.

7.3 Performance analysis and comparison to the literature

7.3.1 MDSDVRP

The results discussed in Section 7.2.2 show that B&C `RYD` is able to solve most instances from set `mdsd40`. This section analyzes its performance on 42 larger instances with up to 249 customers nodes and 5 depots used by Gulczynski et al. (2011) to evaluate their heuristic algorithm. Figure 8, see also Table 4, shows optimality gaps of `RYD` in percent after two hours and compares its objective function values to those from Gulczynski et al. (2011). While `RYD` can solve all 5 instances with $|C| \leq 50$, only one of the larger ones (mdsd2, $|C| = 75$, $|D| = 5$, $D_{\mathrm{R}} = [.7, .9]$) is solved optimally. The

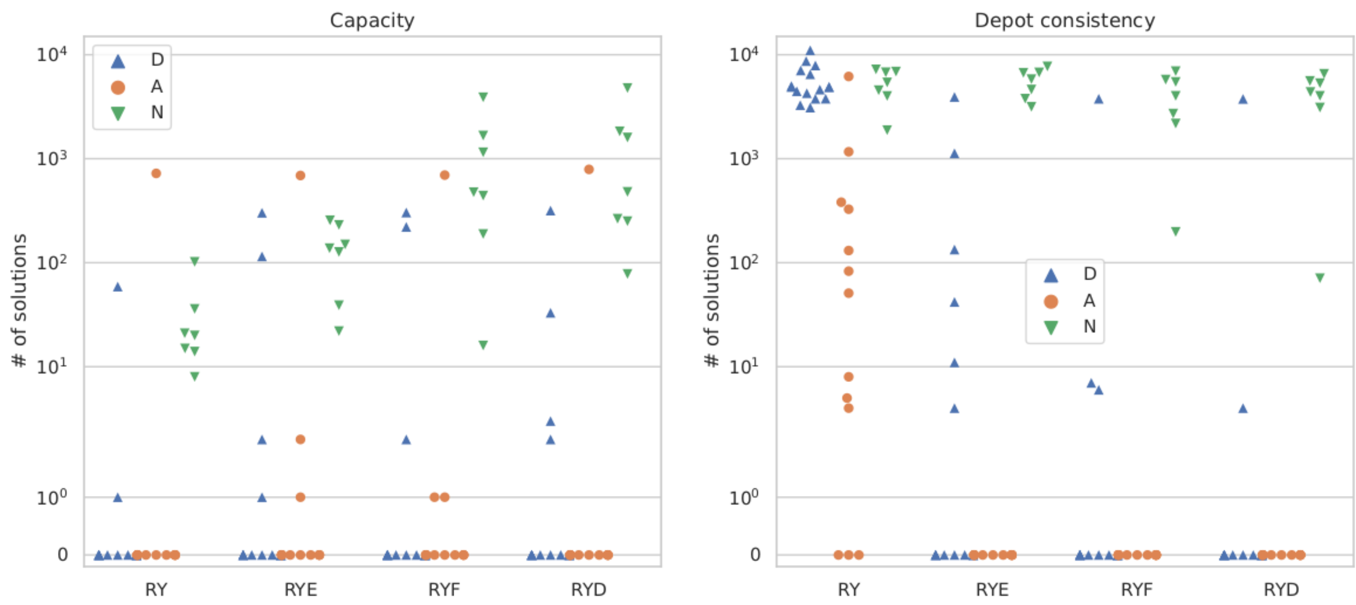

Figure 7: Numbers of infeasible solutions violating capacity and depot-consistency requirements.

relatively small number of decision variables involved and the primal heuristic facilitate, however, the computation of reasonable upper and lower bounds (with gaps of at most 9.1%) even for the largest instances considered. Concerning its primal bounds, we observe that RYD improves upon the solutions reported by Gulczynski et al. (2011) in 39/42 instances and that the initial primal bounds from the heuristic are further improved in 19/42 cases. The detailed results of Table 4 also confirm that all sets of inequalities are violated in the B&C.

7.3.2 MDTSP

Figure 9 (see also Table 6) compares the performance of B&C variant D dynamically separating all sets of depot-consistency inequalities to the best results from the literature (BK) for which we consider the B&C by Bektaş et al. (2017) who used an Intel i7-4790 3.6GHz processor with 8GB RAM, CPLEX 12.6.1 and a time limit of 10,800s (instance set BGS), and the B&C by Benavent and Martínez (2013) who used an Intel i5 2.4GHz processor with 4GB RAM and CPLEX 9 (instance set BM). In these experiments, we test all 26 MDTSP instances from sets BGS and BM which include up to 300 customers and 60 depots. We observe that our B&C significantly outperforms the one by Bektaş et al. (2017), often by orders of magnitude and solves all instances. Besides having a smaller number of variables and less symmetries than theirs (which also works for the asymmetric case), our B&C likely benefits from the fact that our new depot-consistency constraints significantly strengthen the dual bounds. The latter was not observed by Bektaş et al. (2017) for their PF constraints originally referred to as multi-cut inequalities. Comparing our B&C to the B&C by Benavent and Martínez (2013) on instance set BM, we observe that our B&C performs better on the instances with 200 customers and has a similar performance on the relatively easy instances with 100 customers.

7.3.3 SDVRP

We also evaluate the performance of B&C variant RY on the SDVRP. Results are summarized in Table 1 for the four main cases considered in the literature (limited / unlimited fleet, rounded / non-rounded costs). For each case, we report the number of instances (#), number of instances solved by the B&C (RY) and in the literature as well as the number of cases in which our method improves upon (better) or matches (equal) the (previously) best known lower (LB) and upper bound (UB), respectively. Values in parenthesis indicate the number of instances considered in the literature (column #) and the number of instances solved to optimality for the first time by our B&C (column RY). Detailed results can be found in the electronic companion, see Tables 7 to 14. Results indicate that our B&C is competitive to state-of-the-art methods for the SDVRP, despite being developed for a more general setting. It could solve 20 previously unsolved instances and achieve new best lower or upper bounds for a considerable number of cases. The results also confirm the relevance of including explicit single-customer tour variables and yCAPs (9) which are one main methodological difference to similar B&C algorithms proposed for the SDVRP.

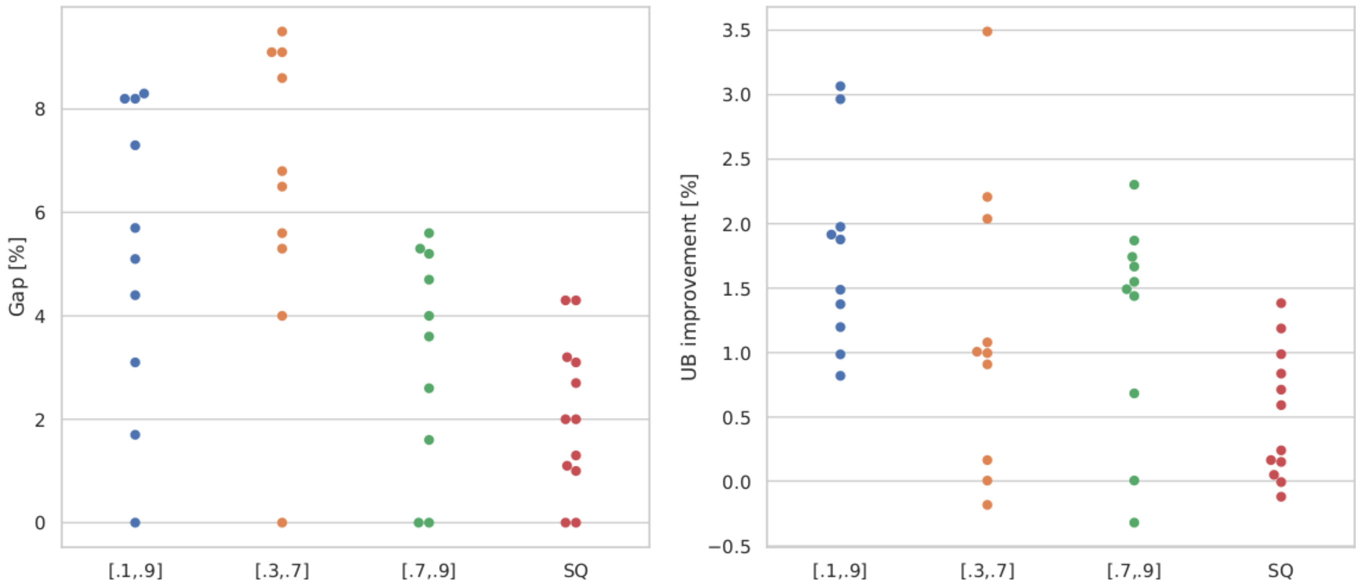

Figure 8: Optimality gaps [%] and improvements of upper bounds [%] over previously best known (Gulczynski et al., 2011) of branch-and-cut for the MDSDVRP.

Table 1: Results for the SDVRP.

			solved		LB		UB	
fleet	cost	#	RY	literature	better	equal	better	equal
limited	rounded	88 (56)	26 (2)	27	43	24	46	30
limited	non-rounded	88 (82)	26 (5)	23	24	21	29	30
unlimited	rounded	88 (56)	27 (7)	23	45	20	45	27
unlimited	non-rounded	88 (82)	27 (6)	23	18	24	26	32
total		352 (276)	106 (20)	96	130	89	146	119

8 Conclusions

We have performed the first study of integer programming formulations and exact solution algorithms of the general case of a vehicle routing problem that combines the two challenging features of having multiple depots and allowing to split node demands over multiple vehicles in an arbitrary way. Our main theoretical contributions are the development and comparison of new sets of valid inequalities for the two challenging subproblems of this MDSDVRP which are to ensure the capacity limits of vehicles and to guarantee that each vehicle returns to its initial depot. In particular, we have proposed a new set of capacity constraints, the yCAPs that exploit single-customer route variables and show that they are not implied by the well-known rounded capacity constraints. The known sets of depot-consistency constraints that aim to ensure that each vehicle returns to its initial depot are considerably enlarged by our work. We show how to adapt path-elimination constraints (Belenguer et al., 2011) to the case of multiple node visits and have generalized them. We also propose sets of (undirected) path-forcing constraints and show that one of these sets dominates all other sets of depot-consistency inequalities considered in this work. Finally, we prove that neither the capacity of vehicles nor the condition that each vehicle must return to its initial depot can be ensured by the inequalities considered in this article.

As the MDSDVRP contains several well-known routing problems as special cases, these developments are of interest from a broader perspective, too. We develop an exact solution framework based on branch-and-cut in which the aforementioned inequalities are separated dynamically and which also includes a primal heuristic and a (practically efficient) feasibility check. The latter identifies infeasible candidate solutions that cannot be eliminated by the inequalities proposed but are instead cut-off by feasibility constraints. Our extensive computational study considers the MDSDVRP and two well-studied special cases, the split-delivery VRP and the multi-depot TSP. Our results show that all inequalities developed in this work contribute positively to the performance of the developed solution method. We observe, in particular, that the dual bounds obtained from including the new set of capacity constraints (yCAPs) are often significantly tighter than the ones when using only rounded capacity constraints. For the MDSDVRP, our results also show that instances up to 50 customers and 4 depots can be solved optimally by the developed solution framework and that its design with a comparably small number of variables ensures that it is applicable to (significantly) larger

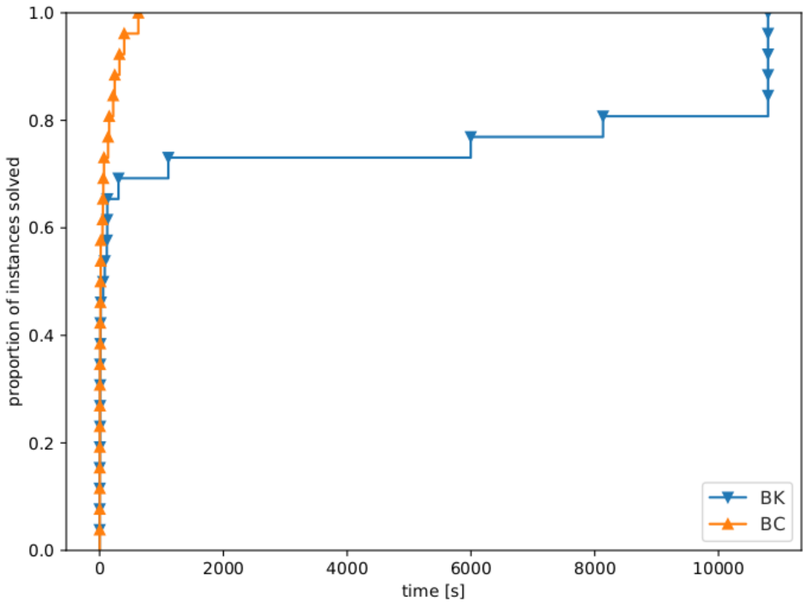

Figure 9: Cumulative proportion of MDTSP instances solved within a given time.

instances for which it computes tight primal and dual bounds. In addition, the results show that our framework significantly outperforms the previous state-of-the-art method for the MDTSP and is competitive to the state-of-the-art for the SDVRP. For the latter, we also provide optimal results for 20 instances that were (to our knowledge) not solved by any prior method. Finally, the results also indicate that the primal heuristic developed performs reasonably well on variants with capacity constraints (and split deliveries), outperforms the MDSDVRP heuristic by Gulczynski et al. (2011) in many cases, but does not perform too well in the uncapacitated case of the MDTSP.

The results and developments of this article raise several questions that could be addressed in future work. Since yCAPs are also valid when the demand of each customer must be satisfied by a single visit of one vehicle, it seems worthwhile to investigate whether they would also significantly strengthen the LP-relaxation bounds for other routing problems. A second interesting question for future research is whether it is possible to derive a set of linear inequalities that guarantee that each vehicle returns to its initial depot without introducing (a significant amount of) additional decision variables. Finally, our computational results indicate that the MDSDVRP is quite challenging which leaves ample possibilities to try developing exact algorithms that can solve larger instances.

Acknowledgments

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Appendix

This appendix is structured as follows: Appendix A contains the proofs of all statements of the main article. Appendix B contains further details concerning the separation of valid inequalities within the branch-and-cut framework. Appendices C to E contain additional and detailed computational results for the MDSDVRP, the MDTSP, and the SDVRP, respectively. Finally, Appendix F analyzes the solution quality of the heuristic used to initialize the B&C algorithm.

A Proofs

A.1 Proofs of Section 1.2 (solution properties)

Proof of Property 4. Assume that there exists a solution satisfying Property 1 in which every route contains at least two split nodes. Let R_1 be one such route containing split nodes s_1 and s_2. Split node s_2 must be contained in at least one additional route, say R_2. Due to Property 1, the latter route must contain a split node s_3 that cannot be contained in a previously considered route, i.e., $s_3 \notin \{s_1, s_2\}$. Continuing the argument one concludes that the last split node encountered in this procedure (say s_ℓ) is contained in only one route which contradicts the fact that s_ℓ is a split node. □

A.2 Proofs of Section 4 (capacity constraints)

Proof of Theorem 1. Let $S' \subseteq S$ be the subset of S such that $q_i < Q$ for each $i \in S'$ and whose nodes in S' are served via single-customer routes, i.e., $\bar{y}_i = 1$, $\forall i \in S'$. Then, inequality (9) can be rewritten as

$$\bar{x}(E(S)) \leq \bar{z}(S) - \frac{q(S \setminus S')}{Q} - |S'|.$$

Since $q_i < Q$ for each $i \in S'$, the demand of each node in S' can be satisfied by one single-customer route. Therefore $\bar{x}(\delta(i)) = 0$, $\forall i \in S'$, $\bar{x}(E(S')) = 0$, and $\bar{z}(S) = \bar{z}(S \setminus S') + |S'|$ which imply that the latter inequality reduces to

$$\bar{x}(E(S \setminus S')) \leq \bar{z}(S \setminus S') - \frac{q(S \setminus S')}{Q}$$

which is a weaker version of (8) for set $S \setminus S'$ and therefore valid for the MDSDVRP. □

Proof of Theorem 2. Consider the candidate solution given in Figure 2a. As discussed in Section 2, this candidate solution may only violate capacity constraints and is indeed infeasible when assuming $Q = 7$, $q_6 = 6$, and $q_i = 2$, $i \in C \setminus \{6\}$. Since no single-customer routes are used, yCAPs (9) are implied by rounded capacity constraints (8) and it therefore remains to show that none of the latter inequalities is violated. To this end, observe that $\lceil q(S)/Q \rceil \leq 2$ for each $S \subseteq C$ and $\bar{z}(S) = |S| + 1$ for each S such that $\bar{x}(E(S)) > 0$. Thus, $\bar{z}(S) - \lceil q(S)/Q \rceil \geq |S| - 1$ holds for each set $S \subseteq C$ and the rounded capacity constraint (8) is therefore dominated by the subtour elimination constraint $\bar{x}(E(S)) \leq |S| - 1$. □

A.3 Proofs of Section 5.1 (path-elimination based constraints)

Proof of Theorem 3. Consider an arbitrary solution $\mathcal{U}$ to the MDSDVRP and let

$$\sum_{j=1}^{k} x(\delta(i_j, I_j)) + y(S) + x(E(S)) \leq z(S) \tag{14}$$

be an arbitrary multi-leg PE inequality for $S \subseteq C$, $i_j \in S$, and $I_j \subseteq D$, $i_j \neq i_p$, $1 \leq j \neq p \leq k$, as defined in (10). We will show that $\mathcal{U}$ satisfies this inequality by induction on the number of vehicle routes from $\mathcal{U}$ visiting at least one node from S.

We observe that all variables values implied by a set of empty routes are equal to zero. In this case, the left- and right-hand side of (14) are equal to zero providing a proper start for our argument. Thus, assume that (14) holds for the variable values associated to an arbitrary subset of the vehicle routes of $\mathcal{U}$. Let $R = (d, u_1, u_2, \ldots, u_\ell, d)$, $d \in D$, $u_i \in C$, $u_i \neq u_j$, $1 \leq i \neq j \leq \ell$, be an arbitrary, not yet added route from $\mathcal{U}$. We will show that (14) holds after adding R.

We first consider the case when R is a single-customer route visiting only one customer, i.e., $\ell = 1$ and $R = (d, u_1, d)$. If $u_1 \in S$, both the left-hand side and the right-hand side of (14) increase by one while neither of the two sides changes if $u_1 \notin S$.

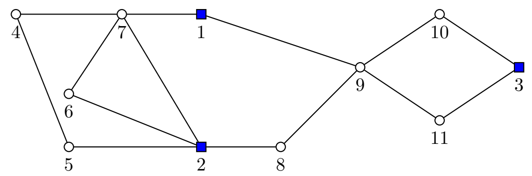

Figure 10: A candidate solution which is infeasible for the MDSDVRP which violates multi-leg PE constraints (10), but satisfies those considering at most two legs.

Next, assume $\ell \geq 2$ and let (V', E') be the subgraph induced by intersection of edges from S and R, i.e., $V' = S \cap \{u_1, u_2, \ldots, u_\ell\}$ and $E' = E(S) \cap \{\{u_i, u_{i+1}\} \mid 1 \leq i < \ell\}$. As depot $d \in R$ is considered in at most one leg of (14), adding route R can increase its left-hand side by at most $1 + |E'|$ while its right-hand side is increased by $|V'|$. The theorem follows since $1 + |E'| \leq |V'|$ holds as R is a route and (V', E') is therefore acyclic. □

Lemma 1 derives a condition under which (10) are redundant that is useful for separation routines and also used in the proof of Theorem 4.

Lemma 1. *Multi-leg PE constraints* (10) *are redundant if the subgraph induced by* $\bar{x}(E(S))$ *is not connected.*

Proof. Proof Consider an arbitrary candidate solution and a set $S \subseteq C$ for which the subgraph induced by $\bar{x}(E(S))$ contains two connected components with node sets $S' \subset S$, $\emptyset \neq S' \neq S$, and $T = S \setminus S'$, i.e., for which $\bar{x}(S', T) = 0$. Let $\bigcup_{j=1}^{k} \delta(i_j, I_j)$ be the legs of an arbitrary multi-leg inequality and assume that $\mathcal{I} = \{i_1, i_2, \ldots, i_\ell\} \subseteq S'$ and $\mathcal{I}' = \{i_1, i_2, \ldots, i_k\} \setminus \mathcal{I} \subseteq T$. Note that the definition of set $\mathcal{I}'$ includes the empty set and that multi-leg inequalities are valid (but redundant) for the case of zero legs (i.e., when $k = 0$). The theorem follows since the multi-leg PE inequality

$$\sum_{j=1}^{k} \bar{x}(\delta(i_j, I_j)) + \bar{y}(S) + \bar{x}(E(S)) \leq \bar{z}(S)$$

is obtained as the sum of the two multi-leg PE constraints

$$\sum_{j: i_j \in \mathcal{I}} \bar{x}(\delta(i_j, I_j)) + \bar{y}(S') + \bar{x}(E(S')) \leq \bar{z}(S')$$

$$\sum_{j: i_j \in \mathcal{I}'} \bar{x}(\delta(i_j, I_j)) + \bar{y}(T) + \bar{x}(E(T)) \leq \bar{z}(T)$$

for S' and T. This holds true since $S = S' \cup T$, $S' \cap T = \emptyset$, ensures that $\bar{y}(S) = \bar{y}(S') + \bar{y}(T)$, $\bar{z}(S) = \bar{z}(S') + \bar{z}(T)$ while $\bar{x}(S', T) = 0$ implies that $\bar{x}(E(S)) = \bar{x}(E(S')) + \bar{x}(E(T))$. □

Proof of Theorem 4. Consider an instance with depots $D = \{1, 2, 3\}$ and customer nodes $C = \{4, 5, \ldots, 11\}$ and the candidate solution to that instance corresponding to the graph given in Figure 10. A solid edge between two nodes u and v indicates $\bar{x}_{uv} = 1$ and we assume the vehicle capacity is not restricting. Clearly, this graph does not represent a feasible solution to the MDSDVRP since the vehicle(s) traversing edges $\{1, 9\}$ and $\{2, 8\}$ cannot return to their initial depot without visiting other depots. We also observe that all nodes have even degree and that the subgraph induced by the customer nodes C is acyclic. Thus, only depot-consistency constraints may be violated.

Indeed, a multi-leg PE inequality (10) is violated for $k = 3$, $I_1 = \{1\}$, $I_2 = \{2\}$, $I_3 = \{3\}$, $i_1 = 9$, $i_2 = 8$, $i_3 = 10$, and $S = \{8, 9, 10\}$, since its left-hand side is equal to five while its right-hand side is equal to four. Observe that no customer node is adjacent to two depots in Figure 10. Thus, for $k = 1$ all multi-leg PE inequalities are satisfied and $\ell := \sum_{j=1}^{k} \bar{x}(\delta(i_j, I_j)) \leq 2$ if $k \leq 2$. Using Lemma 1 and $\bar{y}_i = 0$, $\forall i \in C$, we conclude that $\ell + |S| - 1 > \bar{z}(S)$ and therefore $\bar{z}(S) \leq |S|$ must hold for set $S \subseteq C$ if a multi-leg PE inequality would be violated. Each connected subgraph consisting of customer nodes in Figure 10 that has edges to at least two depots does, however, contain a customer node that is visited twice (i.e., node 7 or 9). Thus, $\bar{z}(S) \geq |S| + 1$ for each relevant customer subset S and the theorem follows. □

Proof of Theorem 5. Consider an instance with depots $D = \{1, 2\}$ and customers $C = \{3, 4, \ldots, 10\}$ and the candidate solution to that instance corresponding to the graph given in Figure 3. A solid edge between nodes u and v indicates $\bar{x}_{uv} = 1$ and we assume that the vehicle capacity is not restricting. We observe that all node degrees are even and the subgraph induced by the customers C is acyclic. All constraints of (1) except depot-consistency inequalities are thus satisfied. It can be easily seen that the graph from Figure 3 does not represent a feasible solution to the MDSDVRP since, e.g., the vehicle traversing edge $\{1, 3\}$ cannot return to depot 1 without visiting depot 2. Also observe that each connected subgraph induced by a set $S \subseteq C$ that is connected to both depots consists of exactly one node with degree four while all other nodes in S have degree two, i.e., $\bar{z}(S) = |S| + 1$. Since $|D| = 2$ and no edge is traversed multiple times the left-hand side value of a PE constraint cannot be larger than $|S| + 1$. Therefore, all multi-leg PE constraints are satisfied. □

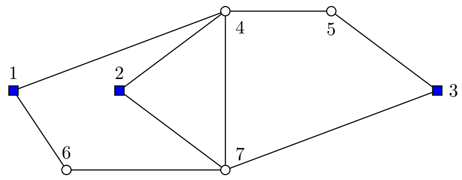

Figure 11: A candidate solution which is infeasible for the MDSDVRP that violates multi-leg PF constraints (12), but does not violate PF constraints (11).

A.4 Proofs of Section 5.2 (path-forcing based constraints)

Proof of Theorem 6. Consider an arbitrary solution of the MDSDVRP and let $S \subseteq C$, $i_j \in S$, $D' = \cup_{j=1}^k I_j$, $I_j \cap I_l = \emptyset$, $1 \leq j \neq l \leq k$, as defined in the multi-leg PF constraints (12). Assume that $\sum_{j=1}^k \bar{x}(i_j, I_j) = \ell$, $\ell \in \mathbb{N}_0$, and recall that (with the exception of single-customer routes) each route traverses each edge at most once. Since each depot is contained in at most one set I_j, $1 \leq j \leq k$, these ℓ edge traversals correspond to ℓ different routes. Each such route enters S by an edge $\{d, i_j\}$, $d \in I_j$, and leaves S via an edge $\{u, v\}$ with $u \in S$, and $v \in (C \cup \{d\}) \setminus S$. If $v = d$, we have $u \neq i_j$ since each edge is traversed only once by each route. Edge $\{u, v\}$ is not contained in one of the legs considered on the left-hand side of (12) since depot d is contained in set I_j and thus in leg $\delta(i_j, I_j)$. Considering all ℓ relevant tours, it follows that

$$\bar{x}(\delta(S, C \setminus S)) + \bar{x}(\delta(S, D')) - \sum_{j=1}^{k} \bar{x}(\delta(i_j, I_j)) \geq \ell.$$

Thus, we have

$$\bar{x}(\delta(S, (C \cup D') \setminus S)) - \sum_{j=1}^{k} \bar{x}(\delta(i_j, I_j)) \geq \sum_{j=1}^{k} \bar{x}(\delta(i_j, I_j))$$

which is equivalent to (12) for the considered solution. □

Proof of Theorem 7. Consider an instance with depots $D = \{1, 2, 3\}$ and customer nodes $C = \{4, 5, 6, 7\}$ and the candidate solution to that instance corresponding to the graph given in Figure 11. A solid edge between nodes u and v indicates $\bar{x}_{uv} = 1$ and we assume that the vehicle capacity is not restricting. As all node-degrees are even and the subgraph induced by the customer nodes C is acyclic, only depot-consistency inequalities may be violated. Observe that (single-leg) PF constraints (11) are not violated since each depot-edge $\{d, i\} \in E_D$, $d \in D$, part of this solution is contained in a cycle starting and ending at the same depot and not passing through any other depot. PF constraints (11) do, however, not ensure that these cycles are edge disjoint. Indeed, the candidate solution is infeasible for the MDSDVRP since the edge $\{4, 7\}$ is contained in all such cycles. It also violates a multi-leg PF inequality (12) for $S = \{4, 5\}$, $D' = \{1, 2, 3\}$ and $k = 2$ with $I_1 = \{1, 2\}$, $I_2 = \{3\}$, $i_1 = 4$, $i_2 = 7$. □

Proof of Theorem 8. We first observe that each multi-leg PF inequality (12) can be rewritten as

$$2 \sum_{j=1}^{k} x(\delta(i_j, I_j)) \leq x(\delta(S)) - x(\delta(S, D \setminus D'))$$

since $x(\delta(S, (C \cup D') \setminus S)) = x(\delta(S)) - x(\delta(S, D \setminus D'))$. Using $x(\delta(S)) = 2z(S) - 2y(S) - 2x(E(S))$ which is obtained by summing up the degree equations (1b) for all customers $i \in S$, it follows that

$$2 \sum_{j=1}^{k} x(\delta(i_j, I_j)) + x(\delta(S, D \setminus D')) \leq 2z(S) - 2x(E(S)) - 2y(S).$$

Rearranging the terms and dividing by two, we obtain

$$\sum_{j=1}^{k} x(\delta(i_j, I_j)) + 0.5x(\delta(S, D \setminus D')) + x(E(S)) + y(S) \leq z(S) \tag{15}$$

which shows that multi-leg PF constraints (12) correspond to lifted versions of multi-leg PE constraints (10) with an additional term $0.5x(\delta(S, D \setminus D'))$ on the left-hand side. This result also implies that there is a one-to-one correspondence between multi-leg PE constraints (10) and the subset of multi-leg PF constraints with $D' = D$. □

Lemma 2 shows that we can focus on sets $S \subseteq C$ for which the subgraph induced by $\bar{x}(E(S))$ is connected when trying to identify violated constraints (12). This result will be used in the proofs of Lemma 3 and Theorem 9.

Lemma 2. *Multi-leg PF constraints* (12) *are redundant if the subgraph induced by* $\bar{x}(E(S))$ *is not connected.*

Proof. Proof Consider an arbitrary candidate solution and a set $S \subseteq C$ for which the subgraph induced by $\bar{x}(E(S))$ contains two disconnected sets $S_1 \subset S$, $\emptyset \neq S_1 \neq S$, and $S_2 = S \setminus S_1$. Let $\bigcup_{j=1}^{k} \delta(i_j, I_j)$ be the legs of an arbitrary multi-leg PF constraint for set S and assume that there exists ℓ, $1 \leq \ell < k$ such that $\{i_1, i_2, \ldots, i_\ell\} \subseteq S_1$ and $\{i_{\ell+1}, \ldots, i_k\} \subseteq S_2$. Consider the two multi-leg PF inequalities

$$2\sum_{j=1}^{\ell} \bar{x}(\delta(i_j, I_j)) \leq \bar{x}(\delta(S_1, (C \cup D_1) \setminus S_1)) \tag{16}$$

$$2\sum_{j=\ell+1}^{k} \bar{x}(\delta(i_j, I_j)) \leq \bar{x}(\delta(S_2, (C \cup D_2) \setminus S_2)) \tag{17}$$

for S_1 and S_2 where $D_1 = \cup_{j=1}^{\ell} I_j$ and $D_2 = \cup_{j=\ell+1}^{k} I_j$ denote the subsets of depots assigned to nodes in S_1 and S_2, respectively. Summing up the latter two inequalities, we obtain

$$2\sum_{j=1}^{k} \bar{x}(\delta(i_j, I_j)) \leq \bar{x}(\delta(S_1, (C \cup D_1) \setminus S_1)) + \bar{x}(\delta(S_2, (C \cup D_2) \setminus S_2)). \tag{18}$$

We also observe that the relations

$$\bar{x}(\delta(S_1, (C \cup D_1) \setminus S_1)) = \bar{x}(\delta(S_1, (C \cup D_1) \setminus (S_1 \cup S_2))) = \bar{x}(\delta(S_1, (C \cup D_1) \setminus S)) \leq \bar{x}(\delta(S_1, (C \cup D) \setminus S))$$
$$\bar{x}(\delta(S_2, (C \cup D_2) \setminus S_2)) = \bar{x}(\delta(S_2, (C \cup D_2) \setminus (S_1 \cup S_2))) = \bar{x}(\delta(S_2, (C \cup D_2) \setminus S)) \leq \bar{x}(\delta(S_2, (C \cup D) \setminus S))$$

hold since $\bar{x}(\delta(S_1, S_2)) = 0$, $D_1 \subseteq D'$ and $D_2 \subseteq D'$. Using above inequalities in (18), we obtain

$$\begin{aligned} 2\sum_{j=1}^{k} \bar{x}(\delta(i_j, I_j)) &\leq \bar{x}(\delta(S_1, (C \cup D_1) \setminus S_1)) + \bar{x}(\delta(S_2, (C \cup D_2) \setminus S_2)) \\ &\leq \bar{x}(\delta(S_1, (C \cup D) \setminus S)) + \bar{x}(\delta(S_2, (C \cup D) \setminus S)) = \bar{x}(\delta(S, (C \cup D) \setminus S)) \end{aligned}$$

which shows that the multi-leg PF inequality for set S is implied by the two inequalities for sets S_1 and S_2. Finally, we note that if $\ell = k$ (which is the case excluded above), the multi-leg PF constraint (16) for set S_1 implies the one for S since $\bar{x}(\delta(S_1, (C \cup D_1) \setminus S_1)) \leq \bar{x}(\delta(S_1, (C \cup D) \setminus S)) \leq \bar{x}(\delta(S, (C \cup D) \setminus S))$. □

The following lemma will be used in the proof of Theorem 9.

Lemma 3. *Consider formulation* (1) *with generic depot-consistency constraints* (1g) *replaced by multi-leg PF constraints* (12). *Assume that a candidate solution contains two customer nodes i and j such that $\bar{x}_{ij} = 1$, $\bar{z}_i = 1$, $\bar{z}_j \geq 2$ and which are both connected to a depot $d \in D$ by an edge traversed exactly once (i.e., $\bar{x}_{di} = 1$ and $\bar{x}_{dj} = 1$). If the considered candidate solution violates a multi-leg PF constraint* (12) *with $i \in S$, then it also violates one for set $S' = S \setminus \{i\}$.*

Proof. Proof Consider nodes $i \in C$, $j \in C$, and $d \in D$, satisfying the conditions in the lemma. Since the inequality (12) for $S = \{i\}$ is not violated, $i \in S$ implies that $j \in S$ (due to Lemma 2). If edge $\{d, i\}$ is not part of the legs considered on the left-hand side of the multi-leg PF inequality, both the left-hand side and the right-hand side value remains identical when removing node i from S, i.e., considering $S' = S \setminus \{i\}$. Similarly, if edge $\{d, i\}$ is contained in one leg of the multi-leg PF inequality, we can instead consider a leg containing edge $\{d, j\}$ which leaves the left-hand side value unchanged and then remove node i from set S as in the first case. □

Proof of Theorem 9. Consider the candidate solution given in Figure 2b. As discussed in Section 2, this candidate solution is not feasible for the MDSDVRP and may only violate depot-consistency constraints. It therefore remains to show that no multi-leg PF constraint (12) is violated. To simplify our arguments, we first observe that due to Lemma 3 we do not need to consider inequalities for which set S contains a customer node in $\{4, 9, 10, 11, 12, 14\}$. Thus, we assume without loss of generality that $S \subseteq S' = \{5, 6, 7, 8, 13\}$ and that subgraph induced by $\bar{x}(E(S))$ is connected, cf. Lemma 2.

We first observe that all multi-leg PF constraints (12) with $k = 1$ and $|D'| = 1$ are satisfied. They correspond to PF constraints (11) which are satisfied since there exists a route starting and ending at depot $d \in D$ that does not visit any other depots for each depot edge $e = \{d, i\} \in E_D$ with $\bar{x}_e = 1$.

Next, we discuss the case when set S contains only one customer node, i.e., $S = \{i\}$, $i \in S'$. Here, we focus on customer nodes $i \in \{6, 7, 8\}$ for which $\bar{x}(\delta_D(i)) = 2$ and assume that $|D'| = 2$ since customers with one adjacent depot are covered by the previous case and customers adjacent to more than two depots do not exist. Thus, the left-hand side of each relevant multi-leg PF constraint is equal to four. Since $\bar{x}(\delta(\{i\}, C \setminus \{i\})) \geq 2$ for each $i \in \{6, 7, 8\}$ and the two depot edges considered on the left-hand side are also contained in the cut $\bar{x}(\delta(\{i\}, (C \cup D') \setminus \{i\}))$, the right-hand side of each such inequality is at least four. Thus, all multi-leg PF constraints with $|S| = 1$ are satisfied.

We now turn our attention to the case $|D'| = 2$ and $S \subseteq S'$ such that $|S| \geq 2$ and the subgraph induced by $\bar{x}(E(S))$ is connected. We assume w.l.o.g. that set S contains nodes i_1 and i_2 (which may be identical) incident to two depot-edges $\{d_1, i_1\}$ and $\{d_2, i_2\}$ which are considered on the left-hand side of (12). We will show that no multi-leg PF constraint can be violated in this case since $\bar{x}(\delta(S, C \setminus S)) \geq 2$ and thus $\bar{x}(\delta(S, (C \cup D') \setminus S)) \geq 4$ holds (because of the two depot edges $\{d_1, i_1\}$ and $\{d_2, i_2\}$) for all relevant sets S. Observe, that $8 \notin S$, since $8 \in S$ would imply that $\{\{8, 10\}, \{8, 11\}\} \subseteq \delta(S, C \setminus S)$. Next, assume that $5 \in S$ for which we discuss two subcases based on the above assumptions of connectivity and $|S| \geq 2$. Node 7 cannot be in S too since $\{5, 7\} \subseteq S$ implies that $\{\{4, 5\}, \{7, 12\}\} \subseteq \delta(S, C \setminus S)$. Node 6 cannot be in S either since $\{5, 6\} \subseteq S$ and $8 \notin S$ (due to the previous result) implies that $\{\{4, 5\}, \{6, 8\}\} \subseteq \delta(S, C \setminus S)$. We conclude that $S \cap \{5, 8\} = \emptyset$ and thus $S \subseteq \{6, 7, 13\}$ which does not contain a connected set of at least two nodes and therefore contradicts the assumptions made in this case.

It remains to show that all multi-leg PF constraints considering all three depots (i.e., $D' = \{1, 2, 3\}$, $\sum_{j=1}^{k} \bar{x}(\delta(i_j, I_j)) = 3$) with $|S| \geq 2$ such that the subgraph induced by $\bar{x}(E(S))$ is connected are satisfied. Here, we observe that the right-hand side of each associated inequality (12) simplifies to $\bar{x}(\delta(S, V \setminus S))$ and that such an inequality is violated if $\bar{x}(\delta(S, V \setminus S)) \leq 5$. Furthermore, $\{7, 13\} \cap S \neq \emptyset$ as depot 3 is connected only to nodes 7 and 13. Node 13 cannot be in set S of a violated multi-leg PF inequality since $13 \in S$ implies $\{8, 13\} \subseteq S$ (since $|S| \geq 2$) and thus $\bar{x}(\delta(S, V \setminus S)) \geq \bar{x}(\delta(\{13, 8\}, V \setminus S)) \geq 7 > 5$ for any $S \in S'$ such that $\{8, 13\} \subseteq S$. Thus, assume that $7 \in S$ and observe that $\bar{x}(\delta(S, V \setminus S)) = 6$ if $S = \{5, 7\}$. Further adding node 6 (i.e., $\{5, 6, 7\} \subseteq S$), we have $\bar{x}(\delta(S, V \setminus S)) \geq \bar{x}(\delta(\{5, 6, 7\}, V \setminus S)) \geq 7 > 5$ for any such $S \in S'$. The theorem follows since $S \cap \{7, 13\} = \emptyset$ which implies that no violated multi-leg PF inequalities can exist in case $D' = 3$. □

B Separation of valid inequalities

B.1 Exact separation of yCAPs (9)

We first observe that for each set $S \subset C$ the cut form

$$x(\delta(S)) + 2y(S) \geq \frac{2}{Q} \left(q(S) + \sum_{i \in S: q_i < Q} (Q - q_i) \cdot y_i \right)$$

of (9) is obtained using the degree constraints (1b) of all customers $i \in S$. Multiplying this inequality with $Q/2$ and re-arranging the terms, we obtain

$$q(C \setminus S) + \sum_{i \in C \setminus S: q_i < Q} (Q - q_i) \cdot y_i + \frac{Q}{2} x(\delta(S)) + Qy(S) \geq q(C) + \sum_{i \in C: q_i < Q} (Q - q_i) \cdot y_i \tag{19}$$

which suggests the following, cut-based separation of yCAPs based on a directed support graph with node set $V \cup \{s, t\}$ where s is the source and t is the target node. For each depot $d \in D$, arc (s, d) is included with infinite capacity to ensure that all depots are outside set S. For each customer $i \in C$, arc (i, t) is added that has capacity $q_i + (Q - q_i) \cdot \bar{y}_i$ if $q_i < Q$ and capacity q_i otherwise. These arcs account for the first two terms on the left-hand side of (19) for customers outside of S. Depot edges $e = \{d, i\} \in \bar{E}_D$, $d \in D$, $i \in C$, are represented by arcs (d, i) with capacity $\frac{Q}{2}\bar{x}_e + Q\bar{y}_i$ if d is the depot from which single-customer routes to i are performed and with capacity $\frac{Q}{2}\bar{x}_e$ otherwise. Finally, we include arcs (i, j) and (j, i) with capacity $\frac{Q}{2}\bar{x}_e$ for each customer edge $e = \{i, j\} \in \bar{E}_C$. Together with aforementioned depot arcs, they account for the last two terms on the left-hand side of (19). A minimum cut from s to t in this graph corresponds to a violated yCAP if its capacity is less than the right-hand side of (19).

B.2 Exact separation of rounded capacity cuts (8)

Formulation (20), inspired by Martinelli et al. (2013), is used for the exact separation of rounded capacity cuts (8). Variables $s_i \in \{0, 1\}$, $i \in C$, indicate customers in S, variables $h_e \in \{0, 1\}$, $e \in \bar{E}_C$, indicate edges within set S, variables $\kappa \in \{0, \dots, K\}$, indicate the number of vehicles needed to serve S, and variable $\rho \in \{0, \dots, Q - 1\}$ indicates the total residual vehicle capacity.

$$\max \quad \sum_{e \in \bar{E}} \bar{x}_e h_e - \sum_{j \in C} \bar{z}_j s_j + \kappa \tag{20a}$$

$$\text{s.t} \quad h_e \leq s_i \qquad \forall i \in C, \forall e = \{i, j\} \in \bar{E}_C \tag{20b}$$

$$\sum_{i \in C} q_j s_j + \rho = Q\kappa \tag{20c}$$

$$s(C) \geq 2 \tag{20d}$$

All solutions with an objective value larger than zero correspond to a violated cut since the objective function (20a) maximizes the violation of an inequality (8). The associated set S is defined by all nodes $i \in C$ such that $\bar{s}_i = 1$. Constraints (20b) guarantee that edge $\{i, j\} \in \bar{E}_C$ can only be in $E(S)$ if both incident nodes are in S. Equation (20c) links the demand of all nodes in S to the number of vehicles required to serve them and inequality (20d) makes sure that set S contains at least two nodes since the case when $|S| = 1$ is covered by (1k).

Even though the MIP is quite sparse, for large instances the runtime can be long. All solutions found by the MIP with an objective value greater than zero are sorted according to their objective value (non-increasing) and considered in an iterative manner. The capacity cut corresponding to each considered solution is added if its node set S is disjoint to those of all such cuts already added for the current candidate solution. Subsequently, we iterate re-solving the MIP while forcing $s_i = 0$ for all customers i already contained in set S of an already added inequality until no more violated cuts are found. For each identified set S, we also check whether a corresponding yCAP constraint (9) is violated in which case we add this constraint, too.

B.3 Separation of multi-leg depot-consistency constraints (10) and (12)

Formulation (21) used for the exact separation of multi-leg PF constraints (12) is based on their subtour form (15). As we detail below, it is also used to separate multi-leg PE constraints (10) after small modifications. Binary variables s_i, $i \in C$, indicate customers in set S, h_e, $e \in \bar{E}_C$, edges within set S, binary variables l_e, $e \in \bar{E}_D$, indicate edges included in a leg, binary variables f_e, $e \in \bar{E}_D$, indicate depot edges connecting a depot in $D \setminus D'$ and a customer in S, and binary variables t_d, $d \in D$, indicate depots in $D \setminus D'$.

$$\max \sum_{e \in \bar{E}_D} (\bar{x}_e l_e + 0.5\bar{x}_e f_e) + \sum_{e \in \bar{E}_C} \bar{x}_e h_e + \sum_{j \in C} (\bar{y}_j - \bar{z}_j) s_j \tag{21a}$$

$$h_e \leq s_i \qquad \forall i \in C, \forall \{i,j\} \in \bar{E}_C \tag{21b}$$

$$l_e \leq s_j \qquad \forall e = \{d,j\} \in \bar{E}_D \tag{21c}$$

$$t_d + l(\delta(d)) \leq 1 \qquad \forall d \in D \tag{21d}$$

$$f_e \leq t_d \qquad \forall e = \{d,j\} \in \bar{E}_D, d \in D \tag{21e}$$

$$f_e \leq s_j \qquad \forall e = \{d,j\} \in \bar{E}_D, j \in C \tag{21f}$$

$$s(C) \geq 2 \tag{21g}$$

All solutions with an objective value larger than zero correspond to a violated cut since the objective function (21a) maximizes the violation of an inequality (12). Constraints (21b) guarantee that edge $\{i, j\} \in \bar{E}_C$ can only be in $E(S)$ if both incident nodes are in S. Similarly, inequalities (21c) make sure that customer j is in S whenever $\{d, j\} \in \bar{E}_D$ is contained in a leg. Constraints (21d) state that each depot is either contained in exactly one leg or in $D \setminus D'$ while (21e) and (21f) are forcing constraints ensuring that depot edges not part of a leg (but counting with a factor of 0.5 in the objective function) are incident to a depot in $D \setminus D'$ and a customer in S. Finally, inequality (21g) make sure that set S contains at least two nodes since multi-leg PF constraints with $|S| = 1$ are equivalent to constraints (3).

As shown in the proof of Theorem 8, multi-leg PE constraints (10) are a special case of multi-leg PF constraints in which $D' = D$. Thus, the variant of formulation (21) obtained by removing variables t_d, $d \in D$, and f_e, $e \in \bar{E}_D$, is used to separate inequalities (10).

C Computational results for the MDSDVRP

This section contains computational results for the MDSDVRP that accompany the figures and discussion in the article.

Table 2 details LP-relaxation gaps [%] of formulation (1) when including different sets of capacity and depot-consistency inequalities. Notation `R` indicates that a variant includes rounded capacity-cuts (8), `Y` indicates inclusion of yCAPs (9), `E2` of two-leg PE constraints, `E` of multi-leg PE constraints (10), `F1` of (single-leg) PF constraints (11), and `F` of multi-leg PF constraints (12).

Table 3 compares different configurations of the B&C algorithm on MDSDVRP instance set `mdsd40`. Their abbreviations indicate whether they consider rounded capacity cuts (`R`), yCAPs (`Y`), (multi-leg) PE constraints (`E`), (multi-leg) PF constraints (`F`), or all depot-consistency constraints (`D`). The lifting detailed in Section 6.2 is applied for all variants including depot-consistency constraints. In addition to solution times (`time`) or optimality gaps (`gap`) [%] after two hours, the table also reports numbers of candidate solutions identified by the feasibility check that violate capacity (`# capacity infeasible`) or depot-consistency (`# depot infeasible`) constraints.

Table 4 provides detailed results for larger MDSDVRP instances. For the best performing B&C variant (`RYD`), the table reports obtained lower bounds (`LB`), upper bounds (`UB`), solution times (`time`) or remaining gaps (`gap`) [%] after

Table 2: LP gaps [%] of MDSDVRP instances (set `mdsd40`) for capacity and depot-consistency constraints.

instance	$\|D\|$	D_{R}	R	RY	RYE2	RYE	RYF1	RYF
mdsd40-1	2	[.1,.9]	2.34	1.94	1.90	1.90	1.94	1.90
		[.3,.7]	3.74	3.01	3.00	3.00	3.01	3.00
		[.7,.9]	1.86	0.75	0.70	0.70	0.75	0.70
	4	[.1,.9]	3.92	3.02	2.68	2.66	2.95	2.65
		[.3,.7]	3.95	2.91	2.81	2.80	2.91	2.79
		[.7,.9]	2.13	0.69	0.59	0.59	0.66	0.58
	6	[.1,.9]	2.66	2.32	2.17	2.09	2.31	2.07
		[.3,.7]	4.90	3.94	3.81	3.81	3.93	3.80
		[.7,.9]	1.33	0.31	0.06	0.05	0.08	0.05
mdsd40-2	2	[.1,.9]	1.99	1.14	1.08	1.08	1.14	1.08
		[.3,.7]	4.28	3.55	3.46	3.46	3.55	3.46
		[.7,.9]	1.75	0.71	0.65	0.65	0.69	0.65
	4	[.1,.9]	2.69	2.08	2.01	2.01	2.08	2.01
		[.3,.7]	5.64	4.34	4.12	4.12	4.34	4.12
		[.7,.9]	1.41	0.29	0.24	0.23	0.25	0.23
	6	[.1,.9]	4.41	3.95	3.45	3.37	3.69	3.37
		[.3,.7]	6.27	5.03	4.58	4.47	4.92	4.46
		[.7,.9]	1.42	0.45	0.32	0.31	0.37	0.31
mdsd40-6	2	[.1,.9]	4.40	3.58	3.58	3.58	3.58	3.58
		[.3,.7]	3.97	2.97	2.97	2.97	2.97	2.97
		[.7,.9]	2.65	1.06	1.05	1.05	1.06	1.05
	4	[.1,.9]	4.34	3.58	3.56	3.54	3.58	3.54
		[.3,.7]	4.48	3.01	2.88	2.83	3.01	2.83
		[.7,.9]	1.71	0.92	0.42	0.42	0.60	0.42
	6	[.1,.9]	4.40	3.55	3.47	3.46	3.54	3.44
		[.3,.7]	3.52	2.52	1.87	1.78	2.32	1.74
		[.7,.9]	0.60	0.41	0.20	0.20	0.22	0.20

two hours, and numbers of cuts added for integer (`# lazy cons`) and fractional (`# fract cuts`) candidate solutions. The latter two categories are reported for symmetry breaking constraints (`SY`), subtour elimination constraints (`SE`), yCAPs (`Y`), rounded capacity constraints (`R`), depot-consistency constraints (`DC`), and feasibility cuts (`F`). In addition, the table also reports best known solutions from the literature (`BK`) and objective values of the heuristic solutions (`heur`) used to initialize the B&C algorithm.

D Computational results for the MDTSP

This section contains detailed computational results for the MDTSP that accompany the figures and discussion in the article. Table 5 details LP-relaxation gaps [%] of formulation (1) when including different sets of depot-consistency inequalities. Notation `B` indicates that a variant includes no such constraints, `E2` indicates inclusion of two-leg PE constraints, `E` of multi-leg PE constraints (10), `F1` of (single-leg) PF constraints (11), and `F` of multi-leg PF constraints (12).

Table 6 reports obtained lower bounds (`LB`), upper bounds (`UB`), solutions times (`time`), and numbers of cuts added for integer (`# lazy cons`) and fractional (`# fract cuts`) candidate solutions for B&C variant (`D`) considering all depot-consistency constraints. The latter two categories are reported for symmetry breaking constraints (`SY`), subtour elimination constraints (`SE`), and depot-consistency constraints (`DC`). In addition, the table also reports best known values (lower bounds, upper bounds, solution times) from the literature (`BK`) reported by Bektaş et al. (2017) and Benavent and Martínez (2013). For the latter, a dash indicates that this instance has not been solved optimally by any of these methods. Finally, objective values of the heuristic solutions (`heur`) used to initialize the B&C algorithm are also provided.

E Computational results for the SDVRP

This section contains detailed computational results for the SDVRP that accompany the figures and discussion in the article. Tables 7 to 14 provide detailed results for SDVRP instances considered in the literature for the cases of a limited and unlimited fleet as well as with and without rounding up all costs to integer values. For the best performing B&C variant `RY`, the table reports obtained lower bounds (`LB`), upper bounds (`UB`), solutions times (`time`) or remaining gaps (`gap`) [%] after two hours, and numbers of cuts added for integer (`# lazy cons`) and fractional (`# fract cuts`) candidate solutions. The latter two categories are reported for symmetry breaking constraints (`SY`), subtour elimination constraints (`SE`), yCAPs (`Y`), rounded capacity constraints (`R`), and feasibility cuts (`F`). These tables also reports best known values (`LB`, `UB`, `time`) from the literature (`BK`) and objective values of the heuristic solutions (`heur`) used to initialize the B&C algorithm. A dash instead of a best known lower or upper bounds indicates that no such bound has been reported in the literature for the corresponding instance.

Table 3: Comparison of B&C configurations for the MDSDVRP on instance set `mdsd40` with $|C| = 40$.

			time (gap)				# capacity infeasible				# depot infeasible			
instance	$\|D\|$	D_{R}	RY	RYE	RYF	RYD	RY	RYE	RYF	RYD	RY	RYE	RYF	RYD
mdsd40-1	2	[.1,.9]	(0.1)	218	176	**158**	0	1	0	0	11017	0	0	0
		[.3,.7]	(0.4)	650	618	**534**	0	0	0	0	7057	42	7	0
		[.7,.9]	159	158	210	**117**	0	0	1	0	5	0	0	0
	4	[.1,.9]	(1.0)	2516	816	**794**	0	0	0	0	4266	0	0	0
		[.3,.7]	367	**171**	176	173	0	0	0	0	131	0	0	0
		[.7,.9]	138	52	**31**	57	0	2	0	0	327	0	0	0
	6	[.1,.9]	(0.5)	275	204	**193**	0	0	0	0	3778	0	0	0
		[.3,.7]	(0.8)	**449**	525	461	0	0	0	0	3769	0	0	0
		[.7,.9]	74	**10**	11	**10**	0	0	0	0	0	0	0	0
mdsd40-2	2	[.1,.9]	78	82	79	**48**	0	0	0	0	0	0	0	0
		[.3,.7]	(0.4)	1159	987	**735**	59	115	221	33	6481	134	0	0
		[.7,.9]	235	95	92	**90**	0	0	0	0	51	0	0	0
	4	[.1,.9]	72	66	100	**63**	0	0	0	0	4	0	0	0
		[.3,.7]	(0.7)	1787	**1720**	2095	0	0	0	0	4452	0	0	0
		[.7,.9]	**74**	219	173	183	0	0	0	0	0	0	0	0
	6	[.1,.9]	(1.6)	292	**261**	408	0	0	0	0	4864	0	0	0
		[.3,.7]	(2.5)	(0.8)	6360	**4933**	0	0	0	0	4914	0	0	0
		[.7,.9]	71	55	**36**	56	0	0	0	0	8	0	0	0
mdsd40-6	2	[.1,.9]	(1.6)	(1.5)	(1.4)	**(0.7)**	102	150	477	480	4559	3160	2180	3105
		[.3,.7]	(1.7)	**(0.6)**	**(0.6)**	**(0.6)**	21	138	189	265	5419	7723	6953	6554
		[.7,.9]	(0.0)	**209**	287	317	0	0	0	0	5002	4	0	0
	4	[.1,.9]	(1.8)	2386	2082	**1801**	0	302	304	317	4621	1127	0	0
		[.3,.7]	(1.3)	1027	**795**	1543	0	0	0	3	4893	11	6	4
		[.7,.9]	(0.6)	84	147	**81**	0	0	0	0	3273	0	0	0
	6	[.1,.9]	5983	414	**409**	697	723	690	696	790	6182	0	0	0
		[.3,.7]	671	114	129	**111**	0	1	1	0	1167	0	0	0
		[.7,.9]	203	40	57	**31**	0	0	0	0	83	0	0	0
mdsd40-9	2	[.1,.9]	(0.8)	(1.0)	(1.4)	**(0.7)**	20	39	442	251	6893	6754	4025	5599
		[.3,.7]	(0.5)	**(0.4)**	**(0.4)**	**(0.4)**	36	231	1152	1834	6780	6703	5478	5329
		[.7,.9]	(0.8)	**(0.4)**	(0.5)	**(0.4)**	14	127	16	78	1884	3781	2715	4385
	4	[.1,.9]	(1.3)	(0.8)	**(0.4)**	(0.5)	8	22	1670	1600	4014	5822	5762	4028
		[.3,.7]	(0.7)	(0.5)	**(0.2)**	**(0.2)**	15	256	3897	4786	7233	4633	198	71
		[.7,.9]	(1.0)	**2119**	2833	2967	0	0	0	0	3101	3925	3770	3759
	6	[.1,.9]	(0.6)	101	85	**63**	0	0	0	0	7877	0	0	0
		[.3,.7]	(1.3)	133	**98**	151	1	2	2	2	8676	0	0	0
		[.7,.9]	100	**51**	70	61	0	0	0	0	382	0	0	0

F Quality of heuristic solutions

This section analyzes the solution quality of the heuristic used to initialize the B&C algorithm, see Section 6.1. Figures 12 and 13 show the heuristic solution values relative to best known upper bounds reported in the literature for the three problem variants considered, see also Appendices C to E for detailed numerical results. From Figure 12 we observe that the heuristic outperforms the one by Gulczynski et al. (2011) in 32/42 cases and is worse in the remaining 8. The largest improvement is 3.3% (mdsd1, $|C| = 50$, $|D| = 4$, $D_{\text{R}} = [.3, .7]$) while the largest deterioration is only 0.5% (mdsd2, $|C| = 75$, $|D| = 5$, $D_{\text{R}} = [.7, .9]$). We also observe that the heuristic performs worse for the MDTSP. The solutions are between 2.2% (BGS, 300-20-1, $|C| = 300$, $|D| = 20$) and 12.4% (BM, 100-10-2, $|C| = 100$, $|D| = 10$) worse than those reported by Bektaş et al. (2017) which are optimal for 21/26 instances. Heuristic results for SDVRP are shown in Figure 13 for limited / unlimited fleet size (`LF`/`UF`) and rounded / non-rounded costs (`RD`/`NRD`). Even though the SDVRP has been studied in many articles, our heuristic improves 52 out of the 267 primal bounds reported in the literature. The maximum improvement is 13% (p04_110, LF-RD). It performs worse in 179 cases with a maximum deterioration of 5.8% (S101D1, UF-NRD). This deterioration is, however, below 2% in 228 cases.

Overall, we conclude, that the heuristic works well on problem variants with split-deliveries or capacity restrictions, but not so much when these properties are not present as, e.g., the MDTSP. It typically produces high-quality solutions fast and will therefore be included as an initialization heuristic in the B&C configuration used to compare our method against the state-of-the-art.

Table 4: Computational results for the MDSDVRP on instances from the literature.

				BK	BC			# lazy cons						# fract cuts					
D_R	instance	$\|C\|$	$\|D\|$	UB	LB	UB	time (gap)	SY	SE	Y	R	DC	F	SY	SE	Y	R	DC	heur
[.1, .9]	mdsd1	50	4	1018.22	**988.03**	**988.03**	**6534**	3	0	0	43	0	19	152	69	614	5306	930	996.23
	mdsd2	75	5	1289.06	**1249.19**	**1271.33**	(1.7)	2	0	0	35	0	1	127	36	1513	5623	698	1271.33
	mdsd3	100	2	2624.41	**2444.27**	**2575.15**	(5.1)	2	3	0	49	3	1	142	176	1417	5677	85	2575.18
	mdsd4	100	2	2393.23	**2222.92**	**2357.63**	(5.7)	5	6	2	63	0	0	176	153	1853	5025	74	2357.63
	mdsd5	100	3	1963.13	**1838.94**	**1924.35**	(4.4)	4	2	0	53	0	0	121	57	1294	6129	127	1924.35
	mdsd6	100	4	1963.68	**1844.85**	**1903.48**	(3.1)	11	1	0	92	2	0	155	97	1767	6600	483	1903.82
	mdsd7	249	2	16096.91	**14628.10**	**15938.10**	(8.2)	0	0	0	0	0	0	282	1166	2520	3849	522	15938.10
	mdsd8	249	3	13258.26	**12068.70**	**13149.50**	(8.2)	0	0	0	0	0	0	258	852	1698	3535	833	13149.50
	mdsd9	249	4	11959.27	**10758.60**	**11730.20**	(8.3)	0	0	0	0	0	0	192	792	2675	9573	1084	11730.20
	mdsd10	249	5	11377.30	**10424.20**	**11241.00**	(7.3)	0	0	0	0	0	0	216	399	1703	4690	1098	11241.00
[.3, .7]	mdsd1	50	4	990.85	**956.27**	**956.27**	**2871**	0	1	0	18	0	491	127	44	432	2831	550	957.91
	mdsd2	75	5	1223.57	**1162.18**	**1210.36**	(4.0)	6	3	0	37	0	0	139	40	2091	9175	1420	1210.36
	mdsd3	100	2	2558.33	**2388.33**	**2554.07**	(6.5)	2	14	1	75	1	0	81	93	919	3627	104	2554.07
	mdsd4	100	2	2336.65	**2178.54**	**2336.47**	(6.8)	10	3	1	94	0	0	107	125	1543	6618	92	2339.05
	mdsd5	100	3	1871.47	**1749.03**	**1852.82**	(5.6)	4	2	1	54	0	0	134	43	1467	5665	157	1852.88
	mdsd6	100	4	1887.48	**1750.92**	**1849.02**	(5.3)	8	2	0	70	0	0	114	61	1256	5509	228	1851.08
	mdsd7	249	2	**16136.07**	**14624.30**	16165.20	(9.5)	0	0	0	0	0	0	225	915	2450	3749	550	16165.20
	mdsd8	249	3	13444.18	**12172.70**	**13322.00**	(8.6)	0	0	0	0	0	0	225	876	1618	3152	707	13322.00
	mdsd9	249	4	12176.61	**10956.70**	**12054.00**	(9.1)	0	0	0	0	0	0	194	722	2875	9955	987	12054.00
	mdsd10	249	5	11831.52	**10519.40**	**11570.40**	(9.1)	0	0	0	0	0	0	212	480	2847	7459	965	11570.40
[.7, .9]	mdsd1	50	4	1344.99	**1325.64**	**1325.64**	**537**	0	0	0	5	0	0	188	20	346	575	347	1329.29
	mdsd2	75	5	1705.98	**1705.84**	**1705.84**	**1437**	2	1	0	16	0	0	251	24	910	1039	682	1714.45
	mdsd3	100	2	**3878.34**	**3750.82**	3890.69	(3.6)	27	20	3	43	5	5	278	417	1337	2771	143	3890.69
	mdsd4	100	2	3525.24	**3362.58**	**3501.13**	(4.0)	16	25	7	87	2	0	443	748	2414	4870	214	3501.16
	mdsd5	100	3	2772.58	**2659.72**	**2731.22**	(2.6)	14	9	1	84	0	0	317	264	2215	4642	267	2743.79
	mdsd6	100	4	2696.47	**2612.24**	**2654.70**	(1.6)	3	9	0	89	0	0	455	470	2606	3348	761	2664.03
	mdsd7	249	2	25502.49	**23620.30**	**24915.40**	(5.2)	0	0	0	0	0	0	528	1175	1711	2921	452	24915.40
	mdsd8	249	3	20915.02	**19410.60**	**20566.40**	(5.6)	0	0	0	0	0	0	467	741	1361	1984	496	20566.40
	mdsd9	249	4	18844.77	**17531.30**	**18516.40**	(5.3)	0	0	0	0	0	0	443	764	1362	2684	722	18516.40
	mdsd10	249	5	17777.76	**16620.20**	**17445.50**	(4.7)	0	0	0	0	0	0	420	602	1317	2479	783	17445.50
SQ	sq1	32	2	1063.08	**1052.58**	**1052.58**	**28**	0	1	0	3	0	5	50	6	87	242	61	1063.71
	sq2	48	3	1601.02	**1578.87**	**1578.87**	**403**	14	3	0	42	11	31	145	38	355	864	349	1603.10
	sq5	64	2	3434.71	**3394.32**	**3428.98**	(1.0)	1	17	0	120	5	295	337	879	2844	3230	480	3434.70
	sq3	64	4	2142.11	**2098.47**	**2126.84**	(1.3)	1	0	0	37	0	8	353	316	3639	3435	2795	2126.84
	sq4	80	5	2684.02	**2622.64**	**2652.15**	(1.1)	7	3	0	32	0	2	368	211	3196	3448	2246	2656.86
	sq9	96	2	7109.71	**6906.32**	**7050.18**	(2.0)	7	14	0	28	4	0	408	2546	3541	4193	206	7053.50
	sq6	96	3	**5142.06**	**5045.43**	5148.10	(2.0)	15	15	0	31	7	0	322	1628	3294	4061	347	5150.53
	sq7	128	4	6869.14	**6680.95**	**6865.55**	(2.7)	20	23	1	34	7	0	422	1777	2952	3839	677	6865.55
	sq10	144	3	**10586.51**	**10242.90**	10586.90	(3.2)	17	67	0	22	2	0	429	573	1411	1989	229	10586.90
	sq8	160	5	8600.60	**8321.34**	**8587.48**	(3.1)	36	16	3	35	10	0	463	1145	1864	2562	711	8587.48
	sq11	192	4	14135.80	**13496.00**	**14101.60**	(4.3)	15	26	0	19	5	0	489	764	1363	1921	507	14101.60
	sq12	240	5	17739.64	**16884.70**	**17634.50**	(4.3)	4	0	0	0	6	0	681	1416	1734	2611	861	17635.20

Table 5: LP gaps [%] of MDTSP instances for different depot-consistency constraints.

type	instance	B	E2	E	F1	F
BGS	100-10-1	1.61	1.48	1.07	0.92	0.92
	100-10-2	0.85	0.80	0.36	0.35	0.35
	100-10-3	3.86	3.81	3.68	3.66	3.66
	100-20-1	1.62	1.37	0.98	0.89	0.89
	100-20-2	2.45	2.24	1.75	1.65	1.65
	100-20-3	4.05	3.60	3.10	2.91	2.91
	100-5-1	1.72	1.59	1.56	1.56	1.56
	100-5-2	0.50	0.41	0.41	0.41	0.41
	100-5-3	2.87	2.85	2.80	2.80	2.80
	200-10-1	0.86	0.86	0.78	0.78	0.78
	200-20-1	1.13	1.10	1.00	1.00	1.00
	200-40-1	2.15	1.90	1.55	1.46	1.46
	300-10-1	1.35	1.27	1.10	1.08	1.08
	300-20-1	1.56	1.39	1.24	1.19	1.19
	300-30-1	1.79	1.52	1.11	1.02	1.02
	300-40-1	2.00	1.71	1.12	0.99	0.99
	300-60-1	2.19	1.82	1.19	1.03	1.03
BM	100-10-1	1.41	1.16	0.84	0.84	0.84
	100-10-2	1.70	1.63	1.51	1.51	1.51
	100-10-3	2.29	1.77	1.41	1.10	1.10
	100-5-1	1.98	1.94	1.80	1.80	1.80
	100-5-2	1.22	1.13	0.64	0.58	0.58
	100-5-3	2.82	2.59	2.37	2.37	2.37
	200-10-1	1.42	1.36	1.25	1.14	1.14
	200-10-2	1.76	1.75	1.72	1.72	1.72
	200-10-3	1.66	1.66	1.66	1.66	1.66

Table 6: Computational results for the MDTSP on instances from the literature.

type	instance	$\|C\|$	$\|D\|$	BK LB	BK UB	BK time	BC LB	BC UB	BC time	# lazy cons SY	# lazy cons SE	# lazy cons DC	# fract cuts SY	# fract cuts SE	# fract cuts DC	heur
BGS	100-5-1	100	5	1569	1569	117	1569	1569	**9**	0	32	1	6	61	20	1674
	100-5-2	100	5	1705	1705	**1**	1705	1705	2	0	2	0	3	34	9	1771
	100-5-3	100	5	1553	1553	58	1553	1553	**14**	0	35	0	10	66	43	1674
	100-10-1	100	10	1524	1524	20	1524	1524	**6**	0	8	1	13	61	72	1648
	100-10-2	100	10	1702	1702	**2**	1702	1702	3	0	5	0	7	34	29	1787
	100-10-3	100	10	1530	1530	124	1530	1530	**13**	0	34	1	12	49	103	1655
	100-20-1	100	20	1509	1509	**5**	1509	1509	18	0	25	1	29	80	229	1631
	100-20-2	100	20	1695	1695	9	1695	1695	**4**	0	14	1	23	38	105	1826
	100-20-3	100	20	1494	1494	14	1494	1494	**10**	0	15	0	22	39	273	1594
	200-10-1	200	10	2266	2266	5997	2266	2266	**60**	0	35	0	10	91	31	2422
	200-20-1	200	20	2250	2250	1110	2250	2250	**53**	0	18	1	19	86	111	2429
	200-40-1	200	40	2211	2211	8134	2211	2211	**244**	0	33	0	44	107	282	2375
	300-10-1	300	10	2675	2826	-	**2700**	**2700**	**219**	0	43	0	18	274	96	2970
	300-20-1	300	20	2666	2870	-	**2687**	**2687**	**320**	0	67	0	24	192	176	2934
	300-30-1	300	30	2664	2717	-	**2679**	**2679**	**155**	0	26	0	30	84	265	2949
	300-40-1	300	40	2662	2693	-	**2676**	**2676**	**397**	0	49	0	35	90	493	2898
	300-60-1	300	60	2650	2729	-	**2665**	**2665**	**625**	0	49	1	66	200	1164	2907
BM	100-5-1	100	5	37943	37943	**13**	37943	37943	17	0	18	0	13	131	69	40861
	100-5-2	100	5	30832	30832	2	30832	30832	2	0	7	0	4	32	33	32986
	100-5-3	100	5	32850	32850	8	32850	32850	**5**	0	15	0	5	50	30	33754
	100-10-1	100	10	39608	39608	8	39608	39608	**4**	0	7	0	8	48	70	41814
	100-10-2	100	10	30641	30641	9	30641	30641	**5**	0	6	0	5	51	18	34440
	100-10-3	100	10	32521	32521	**1**	32521	32521	3	0	7	0	17	43	80	35183
	200-10-1	200	10	53047	53047	123	53047	53047	**47**	0	35	0	15	66	85	57517
	200-10-2	200	10	39663	39663	305	39663	39663	**137**	0	69	1	11	198	31	43179
	200-10-3	200	10	45289	45289	83	45289	45289	**70**	0	64	1	9	149	17	48628

Table 7: Branch-and-cut results for the SDVRP with limited fleet and rounded costs, part 1/2

instance	$\|C\|$	BK LB	BK UB	BK time	BC LB	BC UB	BC time (gap)	# lazy cons SY	# lazy cons SE	# lazy cons Y	# lazy cons R	# lazy cons F	# fract cuts SY	# fract cuts SE	# fract cuts Y	# fract cuts R	heur
S101D1	100	716.00	716.00	3125	716.00	716.00	**2919**	0	5	0	135	412	0	114	142	7265	731.00
S101D2	100	**1337.10**	**1366.00**	-	1276.00	1374.00	(7.1)	7	21	0	44	0	44	148	597	7355	1374.00
S101D3	100	**1832.20**	**1864.00**	-	1773.00	1895.00	(6.4)	3	39	1	71	0	177	1175	1671	6560	1895.00
S101D5	100	**2737.10**	**2770.00**	-	2676.00	2823.00	(5.2)	0	104	0	42	0	0	3018	2062	5741	2823.00
S51D1	50	458.00	458.00	**0**	458.00	458.00	11	0	2	0	2	0	0	8	24	124	458.00
S51D2	50	703.00	703.00	800	703.00	703.00	**266**	0	10	0	84	130	0	118	217	1788	712.00
S51D3	50	935.17	942.00	-	**942.00**	942.00	**1019**	2	2	0	55	0	82	163	209	2876	954.00
S51D4	50	1551.00	1551.00	2102	1551.00	1551.00	**1174**	0	20	0	121	0	0	2309	406	2013	1575.00
S51D5	50	**1325.34**	1328.00	-	1318.00	1328.00	(0.8)	0	20	0	84	0	0	4466	290	3075	1329.00
S51D6	50	**2153.00**	2153.00	**4619**	2148.00	2153.00	(0.2)	1	44	0	167	0	357	8305	946	1860	2177.00
S76D1	75	592.00	592.00	**30**	592.00	592.00	73	0	3	0	26	0	0	20	58	1804	607.00
S76D2	75	**1061.10**	**1082.00**	-	1044.00	1089.00	(4.1)	0	59	0	172	107	0	843	466	9169	1103.00
S76D3	75	**1395.90**	**1420.00**	-	1375.00	1434.00	(4.1)	7	42	0	133	0	279	1797	1449	7712	1434.00
S76D4	75	**2046.10**	**2073.00**	-	2029.00	2097.00	(3.2)	4	34	0	108	0	588	3133	3277	4576	2097.00
SD1	8	228.00	228.00	**0**	228.00	228.00	1	0	0	0	2	0	0	0	2	2	240.00
SD10	64	**2688.00**	2688.00	**2577**	2677.00	2688.00	(0.4)	0	8	0	36	0	0	3736	679	2801	2772.00
SD11	80	13280.00	13280.00	5241	13280.00	13280.00	**107**	0	38	0	11	0	0	272	706	1056	13300.00
SD12	80	7102.36	7315.00	-	**7178.00**	**7216.00**	(0.5)	0	71	0	150	0	0	4431	1760	4891	7235.00
SD13	96	9933.21	-	-	**10023.00**	**10188.00**	(1.6)	0	93	0	53	0	0	3149	2239	5779	10226.00
SD14	120	-	-	-	**10493.00**	**10889.00**	(3.6)	0	69	0	11	0	0	2113	1584	3984	10889.00
SD15	144	-	-	-	**14738.00**	**15230.00**	(3.2)	0	100	0	3	0	0	1355	1951	4177	15230.00
SD16	144	-	-	-	**3384.00**	**3496.00**	(3.2)	0	72	0	137	0	0	1318	2526	6309	3496.00
SD17	160	-	-	-	**26198.00**	**26875.00**	(2.5)	0	72	0	0	0	0	722	1821	3847	26875.00
SD18	160	-	-	-	**13833.00**	**14534.00**	(4.8)	0	35	0	1	0	0	756	1700	3381	14534.00
SD19	192	-	-	-	**19465.00**	**20481.00**	(5.0)	0	18	0	0	0	0	1039	2686	4634	20481.00
SD2	16	708.00	708.00	**0**	708.00	708.00	1	0	0	0	0	0	0	2	25	25	708.00
SD20	240	-	-	-	**38725.80**	**41046.00**	(5.7)	0	0	0	0	0	0	1598	2365	4682	41046.00
SD21	288	-	-	-	**10887.60**	**11655.00**	(6.6)	0	0	0	0	0	0	1043	2259	3628	11655.00
SD3	16	432.00	432.00	**0**	432.00	432.00	1	0	0	0	2	0	0	0	11	11	452.00
SD4	24	630.00	630.00	**0**	630.00	630.00	1	0	0	0	15	0	0	1	38	38	642.00
SD5	32	1392.00	1392.00	11	1392.00	1392.00	**5**	0	0	0	0	0	0	47	156	257	1392.00
SD6	32	832.00	832.00	**1**	832.00	832.00	3	0	11	0	38	0	0	11	107	137	836.00
SD7	40	3640.00	3640.00	**2**	3640.00	3640.00	3	0	0	0	0	0	0	80	192	208	3640.00
SD8	48	5068.00	5068.00	19	5068.00	5068.00	**8**	0	23	0	20	0	0	98	224	251	5068.00
SD9	48	2046.00	2046.00	126	2046.00	2046.00	**80**	0	8	0	45	0	0	367	412	962	2109.00
eil22	21	375.00	375.00	**0**	375.00	375.00	2	0	0	0	0	0	0	2	35	83	375.00
eil23	22	569.00	569.00	**0**	569.00	569.00	1	0	0	0	0	0	0	3	14	21	569.00
eil30	29	510.00	510.00	14	510.00	510.00	**7**	0	5	0	3	15	0	8	39	48	514.00
eil33	32	835.00	835.00	**2**	835.00	835.00	5	0	0	0	0	0	0	17	115	316	835.00
eil51	50	521.00	521.00	36	521.00	521.00	**26**	0	0	0	5	0	0	8	58	791	526.00
eilA101	100	799.80	**814.00**	-	**807.00**	818.00	(1.3)	0	16	0	72	164	0	128	145	9922	819.00
eilA76	75	**807.60**	**818.00**	-	800.00	829.00	(3.5)	0	37	0	183	60	0	581	264	8602	840.00
eilB101	100	**1040.60**	**1061.00**	-	1011.00	1071.00	(5.6)	3	22	0	93	0	29	50	310	2703	1071.00
eilB76	75	**981.40**	**1002.00**	-	958.00	1012.00	(5.3)	0	48	0	79	0	0	871	353	8174	1012.00
eilC76	75	717.80	733.00	-	**724.00**	733.00	(1.2)	0	4	0	67	30	0	170	182	6755	737.00
eilD76	75	666.10	682.00	-	**679.00**	**679.00**	**1670**	0	2	0	35	0	0	64	131	5742	686.00

Table 8: Branch-and-cut results for the SDVRP with limited fleet and rounded costs, part 2/2

		BK			BC			# lazy cons					# fract cuts				
instance	$\|C\|$	LB	UB	time	LB	UB	time (gap)	SY	SE	Y	R	F	SY	SE	Y	R	heur
p01_1030	50	753.00	753.00	6098	753.00	753.00	**570**	2	4	0	14	139	66	35	163	2355	757.00
p01_1050	50	**993.38**	998.00	-	993.00	998.00	(0.5)	1	22	0	172	0	133	1005	421	4362	1013.00
p01_1090	50	1480.00	1480.00	**5368**	1480.00	1480.00	6017	2	45	0	212	1	88	5172	440	2421	1494.00
p01_110	50	458.00	458.00	**1**	458.00	458.00	16	0	1	0	10	0	0	11	23	175	459.00
p01_3070	50	**1473.00**	**1473.00**	**5355**	1460.00	1479.00	(1.3)	2	21	0	88	0	235	5219	810	3543	1487.00
p01_7090	50	**2137.98**	**2142.00**	-	2133.00	2143.00	(0.5)	2	12	0	113	0	335	7466	1285	2183	2166.00
p02_1030	75	1059.86	1157.00	-	**1060.00**	**1117.00**	(5.1)	0	42	0	79	0	0	940	329	6909	1117.00
p02_1050	75	1449.11	-	-	**1451.00**	**1498.00**	(3.1)	0	31	0	49	0	0	3233	660	6253	1498.00
p02_1090	75	2241.79	-	-	**2244.00**	**2305.00**	(2.6)	0	117	0	130	0	23	4915	1430	5051	2305.00
p02_110	75	612.00	612.00	4284	612.00	612.00	**2706**	0	3	0	27	0	9	60	73	9423	619.00
p02_3070	75	**2166.59**	2431.00	-	2154.00	**2224.00**	(3.1)	2	63	0	76	0	238	4974	1833	4969	2232.00
p02_7090	75	3155.61	-	-	**3157.00**	**3252.00**	(2.9)	7	9	0	7	0	214	284	471	1112	3252.00
p03_1030	100	**1379.00**	-	-	1352.00	**1450.00**	(6.8)	11	34	0	65	0	54	139	545	6562	1450.00
p03_1050	100	**1914.80**	-	-	1890.00	**2013.00**	(6.1)	2	51	0	54	0	135	902	1458	4151	2013.00
p03_1090	100	**2994.36**	-	-	2964.00	**3098.00**	(4.3)	24	37	1	50	0	393	3021	2604	3936	3098.00
p03_110	100	740.76	760.00	-	**747.00**	**749.00**	(0.3)	0	4	0	52	49	0	110	98	7328	760.00
p03_3070	100	**2894.75**	-	-	2855.00	**3006.00**	(5.0)	18	66	0	45	0	240	3454	2621	5149	3006.00
p03_7090	100	**4269.98**	-	-	4249.00	**4385.00**	(3.1)	26	74	1	31	0	548	3094	2517	3447	4385.00
p04_1030	150	-	-	-	**1843.00**	**2041.00**	(9.7)	0	106	0	25	0	0	369	1026	2413	2041.00
p04_1050	150	-	-	-	**2646.00**	**2856.00**	(7.4)	0	70	0	6	0	0	511	1269	2769	2856.00
p04_1090	150	-	-	-	**4289.00**	**4573.00**	(6.2)	3	89	0	5	0	106	440	1300	2464	4574.00
p04_110	150	-	-	-	**890.00**	**926.00**	(3.9)	0	29	0	67	0	0	91	198	3752	926.00
p04_3070	150	-	-	-	**4098.00**	**4433.00**	(7.6)	1	58	0	5	0	81	679	1451	2752	4433.00
p04_7090	150	-	-	-	**6125.00**	**6447.00**	(5.0)	23	157	0	12	0	365	536	1072	1963	6447.00
p05_1030	199	-	-	-	**2226.00**	**2475.00**	(10.1)	0	26	0	0	0	0	813	2443	4766	2475.00
p05_1050	199	-	-	-	**3179.00**	**3545.00**	(10.3)	0	30	0	0	0	144	841	1574	2673	3545.00
p05_1090	199	-	-	-	**5182.00**	**5602.00**	(7.5)	0	17	0	0	0	47	1043	1856	3131	5604.00
p05_110	199	-	-	-	**1015.00**	**1089.00**	(6.8)	0	35	0	25	0	0	77	272	2861	1089.00
p05_3070	199	-	-	-	**5069.00**	**5520.00**	(8.2)	0	50	0	0	0	0	721	1749	3093	5523.00
p05_7090	199	-	-	-	**7832.00**	**8213.00**	(4.6)	9	28	0	0	0	584	1272	1323	2494	8213.00
p10_1030	199	-	-	-	**2226.00**	**2475.00**	(10.1)	0	27	0	0	0	0	813	2443	4766	2475.00
p10_1050	199	-	-	-	**3179.00**	**3545.00**	(10.3)	0	33	0	0	0	144	841	1574	2673	3545.00
p10_1090	199	-	-	-	**5183.00**	**5602.00**	(7.5)	0	31	0	0	0	47	1046	1858	3134	5604.00
p10_110	199	-	-	-	**1015.00**	**1089.00**	(6.8)	0	35	0	25	0	0	77	272	2861	1089.00
p10_3070	199	-	-	-	**5069.00**	**5520.00**	(8.2)	0	39	0	0	0	0	718	1747	3090	5523.00
p10_7090	199	-	-	-	**7832.00**	**8213.00**	(4.6)	10	28	0	0	0	584	1272	1323	2494	8213.00
p11_1030	120	-	-	-	**2788.00**	**3009.00**	(7.3)	0	102	0	30	0	0	383	1433	3166	3009.00
p11_1050	120	-	-	-	**4074.00**	**4290.00**	(5.0)	6	117	0	13	0	96	1753	1972	3722	4290.00
p11_1090	120	-	-	-	**6665.00**	**6930.00**	(3.8)	6	77	2	11	0	274	2916	2912	3989	6931.00
p11_110	120	-	-	-	**1014.00**	**1032.00**	(1.7)	0	74	1	365	159	30	730	795	4433	1032.00
p11_3070	120	-	-	-	**6415.00**	**6726.00**	(4.6)	0	116	0	5	0	221	3590	2920	3232	6726.00
p11_7090	120	-	-	-	**9924.00**	**10205.00**	(2.8)	46	82	0	6	0	679	1797	2143	2534	10205.00

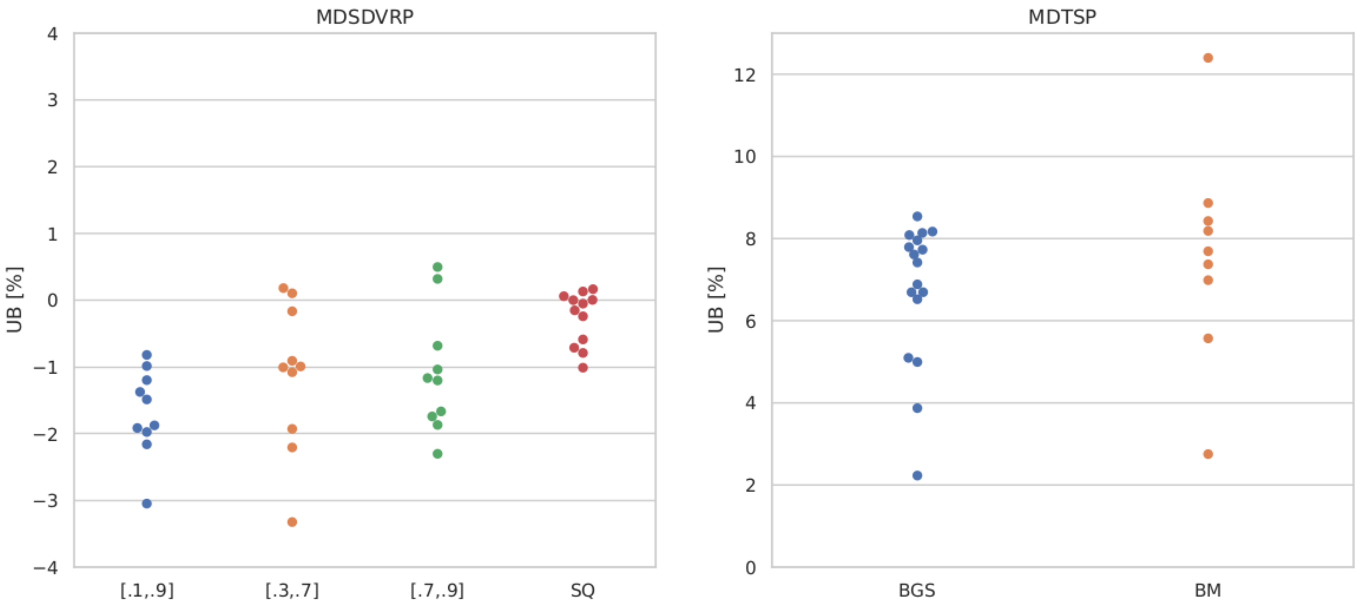

Figure 12: Heuristic solution quality [%] relative to upper bounds of Gulczynski et al. (2011) and Bektaş et al. (2017) for the MDSDVRP and the MDTSP, respectively.

Table 9: Branch-and-cut results for the SDVRP with limited fleet and non-rounded costs, part 1/2

		BK			BC			# lazy cons					# fract cuts				
instance	$\|C\|$	LB	UB	time	LB	UB	time (gap)	SY	SE	Y	R	F	SY	SE	Y	R	heur
S101D1	100	726.59	726.59	**2636**	726.59	726.59	2796	0	17	0	196	108	0	104	111	7038	766.40
S101D2	100	**1358.90**	**1378.43**	-	1298.71	1387.93	(6.4)	3	10	0	107	0	27	50	287	6558	1387.93
S101D3	100	**1853.10**	**1874.81**	-	1786.92	1924.85	(7.2)	9	18	0	96	0	55	108	699	6381	1924.85
S101D5	100	**2767.60**	**2791.22**	-	2712.98	2866.23	(5.3)	0	56	0	116	0	0	360	1384	6623	2866.23
S51D1	50	459.50	459.50	**0**	459.50	459.50	27	0	0	0	4	0	0	7	20	170	461.24
S51D2	50	708.42	708.42	368	708.42	708.42	**146**	0	1	0	24	20	0	22	87	1965	709.63
S51D3	50	940.82	947.97	-	**947.97**	947.97	**1398**	0	1	0	142	0	64	18	128	3435	957.14
S51D4	50	1553.47	1560.88	-	**1560.88**	1560.88	**695**	0	5	0	72	0	0	26	367	2057	1570.79
S51D5	50	1326.61	1333.67	-	**1330.24**	1333.67	(0.3)	0	1	0	169	0	0	35	250	4416	1360.67
S51D6	50	2165.64	2169.10	-	**2169.10**	2169.10	**2233**	1	3	1	93	0	267	223	1060	2473	2191.27
S76D1	75	598.94	598.94	164	598.94	598.94	**57**	0	2	0	15	0	0	12	39	1638	612.22
S76D2	75	**1071.30**	**1087.40**	-	1050.85	1102.03	(4.6)	0	10	0	179	42	0	163	186	10546	1113.00
S76D3	75	**1407.54**	**1427.86**	-	1387.57	1438.24	(3.5)	0	5	0	401	0	250	93	534	10392	1438.24
S76D4	75	**2059.80**	**2079.76**	-	2049.88	2099.94	(2.4)	3	5	0	89	0	375	100	2378	5192	2102.12
SD1	8	228.28	228.28	**0**	228.28	228.28	1	0	0	0	2	0	0	0	2	2	240.65
SD10	64	**2684.88**	**2684.88**	**1671**	2670.17	2684.89	(0.5)	0	31	0	92	0	0	1738	757	3151	2759.67
SD11	80	13280.00	13280.00	5331	13280.00	13280.00	**134**	0	99	0	70	0	0	434	694	1154	13320.00
SD12	80	**7175.80**	**7213.61**	-	7153.52	7230.52	(1.1)	0	74	0	111	0	0	3862	1673	5226	7238.65
SD13	96	**10053.60**	**10110.58**	-	10015.90	10110.60	(0.9)	0	55	0	46	0	0	3428	2066	5566	10187.20
SD14	120	10491.89	**10725.38**	-	**10499.60**	10837.70	(3.1)	0	85	0	21	0	0	2054	1691	4090	10837.70
SD15	144	**14791.26**	**15129.68**	-	14755.80	15379.30	(4.1)	0	86	0	8	0	0	1391	2054	4291	15379.30
SD16	144	3379.33	**3379.33**	**6290**	**3381.08**	3456.45	(2.2)	0	52	0	170	0	0	326	1701	5908	3511.02
SD17	160	26147.89	**26533.39**	-	**26236.40**	26997.60	(2.8)	0	38	0	1	0	0	1103	2095	4563	26997.60
SD18	160	**13896.25**	**14283.51**	-	13813.20	14619.30	(5.5)	0	58	0	6	0	0	724	1523	3164	14619.30
SD19	192	**19592.73**	**20191.20**	-	19424.90	20653.70	(5.9)	0	32	0	0	0	0	1089	2118	3874	20653.70
SD2	16	708.28	708.28	**0**	708.28	708.28	1	0	0	0	0	0	0	4	33	33	708.28
SD20	240	**38916.88**	**39840.00**	-	38724.40	40320.80	(4.0)	0	0	0	0	0	0	1271	2452	4634	40320.80
SD21	288	**11255.32**	**11271.10**	-	10974.80	11579.10	(5.2)	0	27	0	0	0	0	432	1290	2463	11579.10
SD3	16	430.58	430.58	**0**	430.58	430.58	2	0	0	0	0	0	0	0	8	8	430.58
SD4	24	631.05	631.05	**0**	631.05	631.05	3	0	0	0	0	0	0	2	41	41	631.05
SD5	32	1390.57	1390.57	13	1390.57	1390.57	**4**	0	0	0	0	0	0	39	168	285	1390.57
SD6	32	831.24	831.24	**2**	831.24	831.24	4	0	3	0	22	0	0	4	119	237	834.08
SD7	40	3640.00	3640.00	17	3640.00	3640.00	**3**	0	14	0	6	0	0	78	190	221	3640.00
SD8	48	5068.28	5068.28	67	5068.28	5068.28	**12**	0	5	0	9	0	0	103	247	264	5068.28
SD9	48	2044.20	2044.20	147	2044.20	2044.20	**101**	0	6	0	44	0	0	415	364	962	2081.45
eil22	21	375.28	375.28	**0**	375.28	375.28	2	0	0	0	0	0	0	4	34	76	375.28
eil23	22	568.56	568.56	**0**	568.56	568.56	1	0	0	0	0	0	0	2	14	17	568.56
eil30	29	512.72	512.72	**8**	512.72	512.72	9	0	3	0	0	81	0	9	48	66	512.72
eil33	32	837.06	837.06	**2**	837.06	837.06	8	0	1	0	5	20	0	16	105	263	837.06
eil51	50	524.61	524.61	**17**	524.61	524.61	22	0	0	0	8	3	0	16	52	953	527.67
eilA101	100	810.06	**826.14**	-	**816.94**	831.25	(1.7)	0	12	0	104	175	0	98	134	12841	835.79
eilA76	75	**809.67**	**823.89**	-	806.91	827.23	(2.5)	0	15	0	115	121	0	104	161	8258	849.01
eilB101	100	**1055.40**	**1076.26**	-	1023.93	1088.32	(5.9)	3	7	0	124	0	101	39	183	5396	1088.32
eilB76	75	**985.42**	**1009.04**	-	966.46	1019.12	(5.2)	0	4	0	92	0	0	102	241	10511	1019.12
eilC76	75	724.62	**738.67**	-	**729.07**	740.31	(1.5)	0	1	0	32	36	0	67	113	8087	740.31
eilD76	75	677.32	687.60	-	**682.67**	**686.70**	(0.6)	0	3	0	42	22	0	75	50	9188	696.34

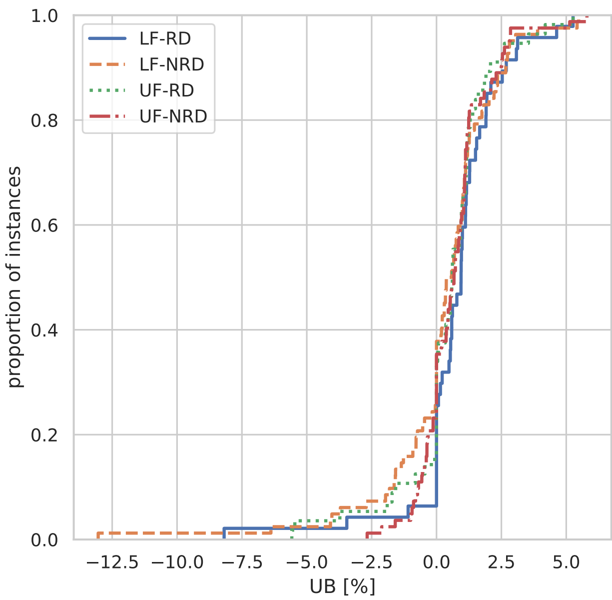

Figure 13: Heuristic solution quality [%] relative to the best known upper bounds for the SDVRP.

Table 10: Branch-and-cut results for the SDVRP with limited fleet and non-rounded costs, part 2/2

		BK			BC			# lazy cons					# fract cuts				
instance	$\lvert C \rvert$	LB	UB	time	LB	UB	time (gap)	SY	SE	Y	R	F	SY	SE	Y	R	heur
p01_1030	50	756.71	756.71	915	756.71	756.71	**343**	0	6	0	35	40	64	21	103	2142	763.04
p01_1050	50	996.93	1005.75	-	**999.21**	1005.75	(0.7)	0	1	0	45	0	60	40	237	5623	1017.76
p01_1090	50	1485.00	1487.18	-	**1487.18**	1487.18	**1326**	0	0	0	94	0	47	23	363	2597	1497.04
p01_110	50	459.50	459.50	**0**	459.50	459.50	11	0	0	0	1	0	0	5	26	231	464.64
p01_3070	50	**1474.10**	1481.71	-	1472.36	1481.71	(0.6)	1	0	1	114	0	157	81	423	5876	1497.66
p01_7090	50	2149.05	2156.14	-	**2152.60**	2156.14	(0.2)	5	3	0	112	0	331	122	1208	2765	2183.33
p02_1030	75	**1093.56**	**1122.91**	-	1064.84	1125.36	(5.4)	0	22	0	180	149	0	131	228	10404	1125.36
p02_1050	75	**1483.17**	1509.79	-	1464.75	**1508.10**	(2.9)	0	2	0	82	0	0	83	365	9418	1509.09
p02_1090	75	**2270.44**	2372.22	-	2263.33	**2327.94**	(2.8)	0	70	0	536	35	21	325	1100	7384	2334.79
p02_110	75	616.58	617.85	-	**617.85**	617.85	**4680**	0	3	0	33	7	9	61	50	10285	635.21
p02_3070	75	**2192.25**	**2235.61**	-	2170.06	2239.89	(3.1)	1	31	2	165	0	246	317	1210	7288	2268.25
p02_7090	75	**3192.10**	**3253.71**	-	3179.68	3262.83	(2.5)	5	54	17	213	0	408	1154	2491	5118	3263.33
p03_1030	100	**1435.23**	1491.82	-	1368.80	**1480.01**	(7.5)	3	2	3	73	0	35	62	297	7127	1480.01
p03_1050	100	**1971.43**	**2018.09**	-	1911.33	2038.20	(6.2)	1	11	1	66	0	50	122	901	7541	2038.20
p03_1090	100	**3043.27**	3136.29	-	3008.80	**3119.57**	(3.6)	11	12	6	92	0	220	156	1236	5457	3119.57
p03_110	100	753.12	762.40	-	**756.25**	**760.00**	(0.5)	0	3	0	54	13	0	98	174	10187	765.25
p03_3070	100	**2945.76**	3044.92	-	2898.70	**3030.84**	(4.4)	15	23	2	104	0	168	476	1539	6089	3030.89
p03_7090	100	**4316.42**	4452.55	-	4302.24	**4417.50**	(2.6)	15	30	17	94	0	531	1167	2306	4129	4417.53
p04_1030	150	**1986.79**	2109.45	-	1886.15	**2052.27**	(8.1)	0	4	0	18	0	0	126	1032	8719	2052.27
p04_1050	150	**2811.64**	2956.18	-	2708.02	**2909.28**	(6.9)	0	7	0	12	0	0	234	1596	9837	2909.28
p04_1090	150	**4474.18**	4708.11	-	4389.91	**4622.66**	(5.0)	2	12	2	10	0	105	298	1724	6615	4622.75
p04_110	150	896.03	1065.94	-	**899.53**	**926.94**	(3.0)	0	4	0	37	0	0	72	90	5449	926.94
p04_3070	150	**4269.77**	4637.69	-	4174.84	**4464.12**	(6.5)	1	36	4	41	0	59	439	1609	4245	4465.86
p04_7090	150	**6287.09**	6529.63	-	6232.93	**6481.48**	(3.8)	19	64	1	53	0	352	538	1657	3253	6481.72
p05_1030	199	**2423.64**	2632.22	-	2259.42	**2524.65**	(10.5)	0	0	0	0	0	0	417	2322	7889	2524.65
p05_1050	199	**3420.17**	3548.13	-	3205.67	**3542.18**	(9.5)	1	32	0	3	0	115	689	1287	2272	3542.18
p05_1090	199	**5425.69**	5894.69	-	5240.85	**5656.48**	(7.3)	0	22	0	0	0	42	780	1875	3113	5656.93
p05_110	199	**1042.37**	1188.45	-	1029.40	**1112.60**	(7.5)	0	12	0	26	0	0	69	233	2715	1112.60
p05_3070	199	**5306.11**	5669.69	-	5138.50	**5557.55**	(7.5)	0	28	0	0	0	0	922	1900	3321	5558.31
p05_7090	199	**8062.24**	8400.74	-	7936.82	**8323.17**	(4.6)	10	25	0	5	0	493	840	1354	2295	8324.13
p10_1030	199	-	-	-	**2259.42**	**2524.65**	(10.5)	0	0	0	0	0	0	417	2322	7889	2524.65
p10_1050	199	-	-	-	**3205.67**	**3542.18**	(9.5)	1	32	0	3	0	115	689	1287	2272	3542.18
p10_1090	199	-	-	-	**5240.85**	**5656.48**	(7.3)	0	32	0	0	0	42	780	1875	3113	5656.93
p10_110	199	-	-	-	**1029.62**	**1112.60**	(7.5)	0	15	0	48	0	0	70	233	2785	1112.60
p10_3070	199	-	-	-	**5138.50**	**5557.55**	(7.5)	0	28	0	0	0	0	922	1900	3321	5558.31
p10_7090	199	-	-	-	**7937.69**	**8323.17**	(4.6)	9	28	0	5	0	493	841	1355	2297	8324.13
p11_1030	120	**2879.63**	**2988.61**	-	2845.77	3012.26	(5.5)	0	16	0	107	0	0	241	1289	6891	3012.26
p11_1050	120	4162.99	**4308.17**	-	**4167.34**	4350.64	(4.2)	0	19	0	86	0	88	252	1916	6532	4350.64
p11_1090	120	**6808.07**	**7020.87**	-	6732.76	7034.31	(4.3)	5	59	3	31	0	192	1344	2354	4681	7034.31
p11_110	120	1023.37	1063.73	-	**1027.54**	**1043.57**	(1.5)	1	58	0	349	105	43	612	740	5196	1049.30
p11_3070	120	**6584.11**	6860.65	-	6537.36	**6748.64**	(3.1)	3	32	10	55	0	128	1089	2183	3951	6748.64
p11_7090	120	**10111.11**	10456.19	-	10065.00	**10322.50**	(2.5)	10	56	8	37	0	428	2547	3216	3628	10322.90

Table 11: Branch-and-cut results for the SDVRP with unlimited fleet and rounded costs, part 1/2

		BK			BC			# lazy cons					# fract cuts				
instance	$\|C\|$	LB	UB	time	LB	UB	time (gap)	SY	SE	Y	R	F	SY	SE	Y	R	heur
S101D1	100	714.87	716.00	-	**716.00**	716.00	**4374**	0	14	0	219	240	5	76	176	6523	746.00
S101D2	100	**1301.93**	**1366.00**	-	1277.00	1374.00	(7.1)	3	16	0	58	0	68	143	677	6362	1374.00
S101D3	100	**1803.51**	**1864.00**	-	1772.00	1890.00	(6.2)	7	47	0	47	0	146	769	1230	5654	1890.00
S101D5	100	**2709.48**	**2770.00**	-	2668.00	2804.00	(4.9)	11	43	0	36	0	343	2804	2590	4994	2804.00
S51D1	50	458.00	458.00	20	458.00	458.00	**14**	0	1	0	9	0	6	9	59	218	470.00
S51D2	50	703.00	703.00	674	703.00	703.00	**198**	0	12	0	50	33	25	45	184	1907	716.00
S51D3	50	933.07	943.00	-	**942.00**	**942.00**	**944**	1	5	0	87	0	99	152	295	2647	954.00
S51D4	50	1547.44	1551.00	-	**1551.00**	1551.00	**2266**	0	22	0	57	0	263	2481	758	2240	1569.00
S51D5	50	**1326.73**	1328.00	-	1316.00	1328.00	(0.9)	0	22	0	97	0	123	4387	350	2948	1329.00
S51D6	50	**2153.00**	**2153.00**	**4110**	2147.00	2166.00	(0.9)	1	46	0	289	0	377	5813	1500	2682	2175.00
S76D1	75	592.00	592.00	134	592.00	592.00	**101**	0	1	0	29	0	3	26	72	2493	607.00
S76D2	75	**1040.67**	**1082.00**	-	1039.00	1092.00	(4.9)	0	21	0	33	0	126	565	486	9695	1092.00
S76D3	75	**1379.57**	**1420.00**	-	1375.00	1434.00	(4.1)	8	44	0	176	0	212	1790	1233	7855	1434.00
S76D4	75	**2034.70**	**2073.00**	-	2030.00	2096.00	(3.1)	7	27	0	73	1	497	3176	3210	4489	2097.00
SD1	8	228.00	228.00	0	228.00	228.00	0	0	0	0	0	0	4	0	21	22	240.00
SD10	64	**2688.00**	**2688.00**	**2317**	2670.00	2698.00	(1.0)	6	19	0	139	0	160	2987	1593	3613	2744.00
SD11	80	13280.00	13280.00	999	13280.00	13280.00	**140**	0	49	0	33	0	202	248	755	1135	13340.00
SD12	80	7133.81	7279.00	-	**7168.00**	**7221.00**	(0.7)	15	46	0	72	0	505	3624	2614	3941	7274.00
SD13	96	9992.94	**10112.00**	-	**10033.00**	10232.00	(1.9)	3	70	0	40	0	781	3202	2627	4167	10232.00
SD14	120	-	-	-	**10485.00**	**10819.00**	(3.1)	24	32	1	17	0	343	2022	1554	3736	10819.00
SD15	144	-	-	-	**14776.00**	**15159.00**	(2.5)	1	61	0	12	0	517	1505	1546	3642	15159.00
SD16	144	-	-	-	**3348.00**	**3491.00**	(4.1)	6	55	0	374	0	217	1038	2446	5347	3491.00
SD17	160	-	-	-	**26269.00**	**26883.00**	(2.3)	7	58	0	1	0	680	970	2052	4122	26883.00
SD18	160	-	-	-	**13826.00**	**14507.00**	(4.7)	8	24	0	6	0	452	785	1528	3212	14507.00
SD19	192	-	-	-	**19418.00**	**20408.00**	(4.9)	1	24	1	1	0	539	963	1835	3516	20408.00
SD2	16	708.00	708.00	**0**	708.00	708.00	1	0	0	0	0	0	31	7	35	46	708.00
SD20	240	-	-	-	**38697.70**	**40657.00**	(4.8)	0	0	0	0	0	979	1453	2153	3949	40657.00
SD21	288	-	-	-	**10956.00**	**11669.00**	(6.1)	2	26	0	0	0	1411	524	2408	2213	11669.00
SD3	16	432.00	432.00	**0**	432.00	432.00	2	0	0	0	0	0	14	0	38	47	432.00
SD4	24	630.00	630.00	**1**	630.00	630.00	3	0	0	0	1	0	32	0	106	114	634.00
SD5	32	1392.00	1392.00	15	1392.00	1392.00	**4**	0	0	0	0	0	50	20	139	259	1392.00
SD6	32	832.00	832.00	**3**	832.00	832.00	4	0	0	0	0	0	39	2	81	198	835.00
SD7	40	3640.00	3640.00	26	3640.00	3640.00	**4**	0	0	0	0	0	77	65	142	210	3640.00
SD8	48	5068.00	5068.00	33	5068.00	5068.00	**11**	0	0	0	0	0	89	100	234	297	5068.00
SD9	48	2046.00	2046.00	174	2046.00	2046.00	**126**	0	0	0	14	0	101	129	254	697	2079.00
eil22	21	375.00	375.00	**1**	375.00	375.00	2	0	0	0	0	0	3	3	35	114	375.00
eil23	22	569.00	569.00	2	569.00	569.00	**1**	0	0	0	0	0	7	2	14	16	569.00
eil30	29	503.00	503.00	**1**	503.00	503.00	3	0	0	0	0	0	7	6	35	38	503.00
eil33	32	835.00	835.00	12	835.00	835.00	**5**	0	0	0	0	0	8	16	118	273	835.00
eil51	50	521.00	521.00	61	521.00	521.00	**20**	0	0	0	3	0	5	9	97	586	526.00
eilA101	100	792.40	**814.00**	-	**806.00**	817.00	(1.3)	0	3	0	51	0	182	113	209	9253	819.00
eilA76	75	792.71	**818.00**	-	**799.00**	823.00	(2.9)	0	19	0	58	0	65	233	264	7321	823.00
eilB101	100	**1017.77**	**1061.00**	-	1012.00	1071.00	(5.5)	2	20	0	112	0	36	66	328	3921	1071.00
eilB76	75	**957.60**	**1002.00**	-	956.00	1015.00	(5.8)	0	42	0	103	0	174	497	322	8796	1015.00
eilC76	75	714.24	**733.00**	-	**721.00**	734.00	(1.8)	0	9	0	60	170	33	150	283	7310	737.00
eilD76	75	667.93	682.00	-	**679.00**	**679.00**	**5746**	0	11	0	106	30	19	154	151	8706	695.00

Table 12: Branch-and-cut results for the SDVRP with unlimited fleet and rounded costs, part 2/2

		BK			BC			# lazy cons					# fract cuts				
instance	$\lvert C \rvert$	LB	UB	time	LB	UB	time (gap)	SY	SE	Y	R	F	SY	SE	Y	R	heur
p01_1030	50	752.00	753.00	-	**753.00**	753.00	**415**	1	12	0	23	156	52	43	120	1981	757.00
p01_1050	50	991.36	998.00	-	**998.00**	998.00	**6710**	0	4	0	19	0	99	597	338	4837	1013.00
p01_1090	50	1480.00	1480.00	3725	1480.00	1480.00	**3655**	1	20	0	127	0	176	2335	493	2374	1493.00
p01_110	50	458.00	458.00	**10**	458.00	458.00	17	0	1	0	3	0	4	10	39	194	458.00
p01_3070	50	**1473.00**	**1473.00**	**6953**	1458.00	1479.00	(1.4)	5	6	0	68	0	258	4305	694	3289	1482.00
p01_7090	50	**2134.87**	2142.00	-	2133.00	2142.00	(0.4)	3	9	0	79	0	415	7553	1360	2350	2166.00
p02_1030	75	**1062.36**	1172.00	-	1061.00	**1108.00**	(4.2)	0	21	0	49	0	63	817	493	8560	1108.00
p02_1050	75	**1456.12**	1557.00	-	1451.00	**1499.00**	(3.2)	0	29	0	28	0	108	2020	828	6248	1499.00
p02_1090	75	**2258.51**	**2304.00**	-	2246.00	2305.00	(2.6)	3	44	0	40	0	509	4320	2833	4796	2305.00
p02_110	75	609.19	612.00	-	**612.00**	612.00	**3254**	0	2	0	18	5	25	51	177	8608	619.00
p02_3070	75	**2177.37**	2253.00	-	2153.00	**2234.00**	(3.6)	3	30	0	57	0	349	4080	2022	4694	2243.00
p02_7090	75	**3171.95**	3233.00	-	3163.00	**3230.00**	(2.1)	8	94	0	125	0	525	4357	2495	3618	3244.00
p03_1030	100	**1379.18**	1542.00	-	1357.00	**1456.00**	(6.8)	6	41	0	23	0	98	131	670	6428	1456.00
p03_1050	100	**1925.21**	2053.00	-	1889.00	**2014.00**	(6.2)	1	36	0	20	0	149	1060	1336	4616	2014.00
p03_1090	100	**3005.48**	3152.00	-	2965.00	**3098.00**	(4.3)	18	45	0	75	0	435	3183	2695	3927	3098.00
p03_110	100	737.64	762.00	-	**744.00**	**750.00**	(0.8)	0	27	0	167	326	22	170	195	8206	789.00
p03_3070	100	**2909.22**	3049.00	-	2860.00	**3001.00**	(4.7)	23	61	0	30	0	273	3173	2413	4579	3001.00
p03_7090	100	**4251.65**	4404.00	-	4245.00	**4368.00**	(2.8)	16	65	3	39	0	563	4080	3289	4127	4368.00
p04_1030	150	-	-	-	**1846.00**	**2034.00**	(9.2)	1	23	0	5	0	86	421	1791	4770	2034.00
p04_1050	150	-	-	-	**2639.00**	**2860.00**	(7.7)	1	31	0	4	0	104	580	1735	3714	2860.00
p04_1090	150	-	-	-	**4299.00**	**4573.00**	(6.0)	1	47	0	0	0	192	448	1386	2617	4573.00
p04_110	150	-	-	-	**889.00**	**947.00**	(6.1)	0	23	0	108	0	13	39	203	2344	947.00
p04_3070	150	-	-	-	**4062.00**	**4416.00**	(8.0)	1	61	0	2	0	158	464	1289	2519	4416.00
p04_7090	150	-	-	-	**6113.00**	**6420.00**	(4.8)	26	68	0	15	0	425	1361	1711	3166	6420.00
p05_1030	199	-	-	-	**2194.51**	**2480.00**	(11.5)	0	0	0	0	0	75	415	2535	5168	2480.00
p05_1050	199	-	-	-	**3178.00**	**3510.00**	(9.5)	13	21	0	0	0	138	980	1419	2388	3510.00
p05_1090	199	-	-	-	**5167.00**	**5579.00**	(7.4)	0	9	0	0	0	213	1084	1911	3150	5580.00
p05_110	199	-	-	-	**1015.00**	**1126.00**	(9.9)	1	26	0	23	0	32	63	336	3186	1126.00
p05_3070	199	-	-	-	**5041.00**	**5484.00**	(8.1)	0	33	0	0	0	210	924	1711	2948	5486.00
p05_7090	199	-	-	-	**7807.00**	**8206.00**	(4.9)	15	25	0	0	0	589	741	1233	2036	8206.00
p10_1030	199	-	-	-	**2194.51**	**2480.00**	(11.5)	0	0	0	0	0	75	415	2535	5168	2480.00
p10_1050	199	-	-	-	**3178.00**	**3510.00**	(9.5)	11	21	0	0	0	138	980	1419	2388	3510.00
p10_1090	199	-	-	-	**5167.00**	**5579.00**	(7.4)	3	19	0	0	0	213	1087	1913	3152	5580.00
p10_110	199	-	-	-	**1015.00**	**1126.00**	(9.9)	0	23	0	17	0	32	63	336	3186	1126.00
p10_3070	199	-	-	-	**5041.00**	**5484.00**	(8.1)	0	27	0	0	0	210	924	1711	2948	5486.00
p10_7090	199	-	-	-	**7807.00**	**8206.00**	(4.9)	14	26	0	0	0	589	741	1233	2036	8206.00
p11_1030	120	-	-	-	**2791.00**	**2976.00**	(6.2)	0	82	0	13	0	62	351	1726	3793	2976.00
p11_1050	120	-	-	-	**4089.00**	**4296.00**	(4.8)	1	200	0	7	0	124	1349	1542	3690	4296.00
p11_1090	120	-	-	-	**6673.00**	**6982.00**	(4.4)	5	111	0	8	0	341	3427	3127	3567	6982.00
p11_110	120	-	-	-	**1013.00**	**1032.00**	(1.8)	0	103	0	407	70	54	1136	906	4681	1032.00
p11_3070	120	-	-	-	**6436.00**	**6663.00**	(3.4)	19	88	1	5	0	188	1712	2106	2883	6663.00
p11_7090	120	-	-	-	**9922.00**	**10193.00**	(2.7)	17	96	1	7	0	726	2876	2731	3028	10193.00

Table 13: Branch-and-cut results for the SDVRP with unlimited fleet and non-rounded costs, part 1/2

		BK			BC			# lazy cons					# fract cuts				
instance	$\|C\|$	LB	UB	time	LB	UB	time (gap)	SY	SE	Y	R	F	SY	SE	Y	R	heur
S101D1	100	716.92	726.59	-	**726.59**	726.59	**3318**	0	22	0	191	110	7	138	122	5419	768.72
S101D2	100	**1356.78**	**1378.43**	-	1296.13	1394.67	(7.1)	0	6	0	56	0	31	72	406	6711	1394.67
S101D3	100	**1845.07**	**1874.81**	-	1784.59	1921.54	(7.1)	0	4	1	75	0	89	82	718	7063	1921.59
S101D5	100	**2758.21**	**2791.22**	-	2687.48	2816.49	(4.6)	8	8	1	66	0	87	149	1007	5854	2816.86
S51D1	50	459.50	459.50	**13**	459.50	459.50	17	0	0	0	13	0	5	9	40	195	472.61
S51D2	50	708.42	708.42	2376	708.42	708.42	**248**	0	3	0	25	8	17	24	129	2263	711.42
S51D3	50	947.97	947.97	20738	947.97	947.97	**1486**	0	0	0	105	0	72	25	147	3195	957.14
S51D4	50	1560.88	1560.88	**3827**	1560.88	1560.88	3860	0	3	0	67	9	153	31	505	3232	1569.81
S51D5	50	**1333.67**	1333.67	**6310**	1329.62	1333.67	(0.3)	2	1	0	200	0	58	48	355	4544	1348.19
S51D6	50	2169.10	2169.10	16755	2169.10	2169.10	**6414**	7	7	4	87	0	304	322	941	2273	2191.75
S76D1	75	598.94	598.94	563	598.94	598.94	**88**	0	2	0	11	0	3	18	79	1744	613.99
S76D2	75	**1066.88**	**1087.40**	-	1044.45	1098.92	(5.0)	0	3	0	86	0	38	89	266	10587	1098.92
S76D3	75	**1406.85**	**1427.86**	-	1385.89	1438.24	(3.6)	2	2	0	132	0	257	97	445	10349	1438.24
S76D4	75	**2053.66**	**2079.76**	-	2046.84	2088.79	(2.0)	1	1	0	206	0	478	132	1868	5863	2101.76
SD1	8	228.28	228.28	0	228.28	228.28	0	0	0	0	0	0	4	0	19	18	240.00
SD10	64	**2684.86**	2684.88	**1903**	2667.57	2684.88	(0.6)	4	11	0	36	0	202	2034	1195	3503	2746.57
SD11	80	13280.00	13280.00	1340	13280.00	13280.00	**122**	0	23	0	19	0	232	200	655	1069	13340.00
SD12	80	7135.27	7270.87	-	**7159.98**	**7242.17**	(1.1)	15	25	1	75	0	433	3984	2298	5028	7242.17
SD13	96	9992.74	**10110.58**	-	**10035.20**	10110.60	(0.7)	4	54	0	47	0	582	2840	2362	4695	10230.20
SD14	120	**10502.76**	**10754.70**	-	10487.90	10849.10	(3.3)	34	32	1	19	0	369	1931	1833	4005	10849.10
SD15	144	**14787.05**	**15151.10**	-	14755.40	15342.60	(3.8)	14	31	1	7	0	501	1302	2002	3907	15342.60
SD16	144	**3379.33**	**3379.33**	**9772**	3349.78	3467.84	(3.4)	10	29	1	28	0	143	252	1134	3707	3467.84
SD17	160	26166.80	**26547.44**	-	**26226.00**	26798.50	(2.1)	8	36	0	0	0	669	1156	2031	4093	26798.50
SD18	160	**13892.74**	**14334.03**	-	13808.00	14438.30	(4.4)	21	53	0	7	0	492	750	1467	2965	14438.30
SD19	192	**19584.84**	**20191.20**	-	19457.50	20562.10	(5.4)	2	20	0	0	0	556	1503	1932	3755	20562.10
SD2	16	708.28	708.28	**0**	708.28	708.28	1	0	0	0	0	0	28	8	27	37	708.28
SD20	240	**38901.37**	**39840.00**	-	38734.90	40375.30	(4.1)	0	0	0	0	0	913	1497	2189	4182	40375.30
SD21	288	**11254.83**	**11271.10**	-	10971.40	11592.60	(5.4)	0	0	0	0	0	919	721	2066	3099	11592.60
SD3	16	430.58	430.58	**0**	430.58	430.58	2	0	0	0	0	0	15	0	36	54	430.58
SD4	24	631.05	631.05	**1**	631.05	631.05	4	0	0	0	0	0	30	0	73	85	633.44
SD5	32	1390.57	1390.57	51	1390.57	1390.57	**5**	0	0	0	0	0	46	16	194	298	1390.57
SD6	32	831.24	831.24	4	831.24	831.24	4	0	0	0	0	0	36	2	144	291	831.24
SD7	40	3640.00	3640.00	55	3640.00	3640.00	**4**	0	0	0	0	0	90	58	174	226	3640.00
SD8	48	5068.28	5068.28	72	5068.28	5068.28	**7**	0	24	0	9	0	97	103	255	335	5068.28
SD9	48	2044.20	2044.20	267	2044.20	2044.20	**114**	0	7	0	29	0	108	184	278	902	2078.86
eil22	21	375.28	375.28	6	375.28	375.28	**2**	0	0	0	0	0	4	3	43	109	375.28
eil23	22	568.56	568.56	1	568.56	568.56	1	0	0	0	0	0	9	3	14	18	568.56
eil30	29	505.01	505.01	**2**	505.01	505.01	3	0	0	0	0	0	3	10	42	55	505.01
eil33	32	837.06	837.06	46	837.06	837.06	**7**	0	1	0	6	20	10	15	107	320	837.06
eil51	50	524.61	524.61	95	524.61	524.61	**16**	0	0	0	4	0	2	7	68	729	527.67
eilA101	100	804.40	**826.14**	-	**815.00**	835.79	(2.5)	0	9	0	109	102	123	115	221	11076	835.79
eilA76	75	**809.58**	**823.89**	-	805.25	828.75	(2.8)	0	7	0	107	31	32	96	166	9369	834.12
eilB101	100	**1055.59**	**1076.26**	-	1023.72	1088.19	(5.9)	1	9	0	137	0	117	48	169	5076	1088.19
eilB76	75	**984.13**	**1009.04**	-	962.02	1015.83	(5.3)	0	6	0	125	1	70	131	229	9675	1015.83
eilC76	75	722.76	738.67	-	**728.73**	738.67	(1.3)	0	5	0	77	57	19	79	130	9574	747.83
eilD76	75	674.17	**687.60**	-	**681.31**	690.17	(1.3)	0	1	0	39	7	13	80	123	10800	692.66

Table 14: Branch-and-cut results for the SDVRP with unlimited fleet and non-rounded costs, part 2/2

		BK			BC			# lazy cons					# fract cuts				
instance	$\|C\|$	LB	UB	time	LB	UB	time (gap)	SY	SE	Y	R	F	SY	SE	Y	R	heur
p01_1030	50	756.70	756.71	20494	**756.71**	756.71	**281**	0	2	0	38	3	55	14	111	2126	763.04
p01_1050	50	**1005.75**	1005.75	**23257**	998.04	1005.75	(0.8)	7	2	0	106	0	59	65	173	5986	1016.84
p01_1090	50	1487.18	1487.18	7155	1487.18	1487.18	**1388**	0	0	0	54	0	105	25	330	3000	1494.88
p01_110	50	459.50	459.50	**8**	459.50	459.50	12	0	0	0	8	0	5	10	51	230	464.64
p01_3070	50	**1481.71**	1481.71	**12418**	1473.52	1481.71	(0.6)	0	1	0	128	0	168	61	432	6421	1497.66
p01_7090	50	2155.80	2155.80	42587	2155.80	2155.80	**7098**	1	0	0	32	0	262	94	783	2446	2179.23
p02_1030	75	**1095.65**	1121.82	-	1067.52	**1119.73**	(4.7)	0	22	0	164	136	52	163	265	10679	1124.12
p02_1050	75	**1482.50**	1514.39	-	1461.17	**1509.02**	(3.2)	0	4	0	81	0	88	98	510	9764	1509.02
p02_1090	75	**2272.05**	**2318.28**	-	2259.26	2329.53	(3.0)	2	13	0	126	0	416	238	1766	6292	2330.61
p02_110	75	612.45	617.85	-	**617.85**	617.85	**5415**	0	5	0	32	9	26	91	153	10496	635.21
p02_3070	75	**2190.16**	**2237.19**	-	2172.76	2253.51	(3.6)	4	11	0	86	0	224	278	1097	9406	2256.34
p02_7090	75	**3192.55**	**3232.15**	-	3181.27	3237.92	(1.7)	4	26	11	116	0	401	807	1748	4711	3253.97
p03_1030	100	**1437.78**	**1477.35**	-	1369.50	1480.01	(7.5)	3	7	0	90	0	40	62	484	7291	1480.01
p03_1050	100	**1971.34**	2040.92	-	1898.68	**2038.20**	(6.8)	2	19	0	89	0	79	105	1074	7349	2038.20
p03_1090	100	**3042.93**	3127.06	-	3001.04	**3115.42**	(3.7)	23	19	3	88	0	322	276	1740	5668	3115.54
p03_110	100	749.42	769.42	-	**759.38**	**760.00**	(0.1)	0	3	0	28	0	3	82	126	7885	761.77
p03_3070	100	**2945.42**	3030.66	-	2896.12	**3026.71**	(4.3)	15	18	2	106	0	275	314	1657	6422	3026.71
p03_7090	100	**4334.44**	**4417.57**	-	4305.20	4436.99	(3.0)	23	35	16	93	0	422	828	2186	4192	4437.00
p04_1030	150	**1986.34**	2066.46	-	1876.92	**2056.87**	(8.7)	0	1	0	9	0	61	122	1021	8821	2056.87
p04_1050	150	**2811.98**	2917.80	-	2699.47	**2901.97**	(7.0)	0	0	0	0	0	84	274	1770	10886	2901.97
p04_1090	150	**4474.92**	4665.87	-	4388.76	**4624.60**	(5.1)	6	17	0	5	0	181	321	1558	5402	4624.60
p04_110	150	**895.46**	**947.14**	-	895.02	966.90	(7.4)	0	14	0	121	0	15	33	137	2419	966.90
p04_3070	150	**4267.33**	**4438.76**	-	4153.56	4455.14	(6.8)	7	14	4	8	0	166	397	1487	4494	4455.14
p04_7090	150	**6284.76**	6482.19	-	6216.65	**6457.37**	(3.7)	23	46	2	16	0	365	485	1746	3598	6458.23
p05_1030	199	**2423.99**	2583.29	-	2254.17	**2528.69**	(10.9)	0	0	0	0	0	72	359	2236	7792	2528.69
p05_1050	199	**3420.23**	3568.25	-	3244.54	**3542.18**	(8.4)	0	0	0	0	0	121	660	2911	9022	3542.18
p05_1090	199	**5422.95**	5673.18	-	5257.30	**5627.28**	(6.6)	0	27	0	1	0	182	1033	1832	3096	5627.35
p05_110	199	**1042.37**	1148.27	-	1028.84	**1117.58**	(7.9)	0	16	0	28	0	22	73	222	2828	1117.58
p05_3070	199	**5304.09**	5559.29	-	5147.61	**5505.91**	(6.5)	0	14	0	6	0	181	1025	1867	3339	5506.85
p05_7090	199	**8062.14**	8359.65	-	7937.06	**8301.59**	(4.4)	2	22	0	1	0	538	1117	1500	2584	8301.59
p10_1030	199	-	-	-	**2254.17**	**2528.69**	(10.9)	0	0	0	0	0	72	359	2236	7792	2528.69
p10_1050	199	-	-	-	**3244.54**	**3542.18**	(8.4)	0	0	0	0	0	121	660	2911	9022	3542.18
p10_1090	199	-	-	-	**5257.30**	**5627.28**	(6.6)	0	22	0	1	0	182	1033	1832	3096	5627.35
p10_110	199	-	-	-	**1028.68**	**1117.58**	(8.0)	0	15	0	24	0	21	71	222	2735	1117.58
p10_3070	199	-	-	-	**5147.73**	**5505.94**	(6.5)	0	12	0	8	0	181	1025	1867	3339	5506.90
p10_7090	199	-	-	-	**7937.06**	**8301.59**	(4.4)	5	25	0	0	0	538	1117	1500	2584	8301.59
p11_1030	120	**2867.79**	**2983.82**	-	2844.93	3017.20	(5.7)	0	11	0	90	0	41	185	1665	6082	3017.20
p11_1050	120	4156.68	**4259.94**	-	**4168.03**	4350.64	(4.2)	0	9	0	111	0	72	270	1764	6234	4350.64
p11_1090	120	**6780.19**	**6995.85**	-	6738.52	7173.25	(6.1)	12	38	4	41	0	236	1059	1815	3518	7173.58
p11_110	120	1023.39	1055.28	-	**1024.49**	**1049.30**	(2.4)	0	44	0	572	18	50	565	744	3892	1049.30
p11_3070	120	**6593.28**	6822.31	-	6542.85	**6711.40**	(2.5)	12	34	5	38	0	138	839	1712	3685	6713.78
p11_7090	120	**10113.55**	10376.94	-	10111.30	**10342.90**	(2.2)	9	53	9	49	0	576	1446	2871	3418	10343.40